Metric Properties of Attractors for Vector Fields via Bounded, Nonautonomous Control

A DISSERTATION
SUBMITTED TO THE FACULTY OF THE
UNIVERSITY OF MINNESOTA
BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

May, 2019

Acknowledgements

Thanks go firstly to Dick McGehee, a steadfast advisor who has given generously of his time and ideas to support this work. Dick inspired me to harness not only The Force of Analysis but also the force of my own interests and strengths to pursue a career in mathematics. It has been a privilege to learn "how to think about things" from him, including his holistic view of the connections between applied and abstract mathematics.

The McGehee research group helped this project move forward in many ways. Bill Satzer raised insightful questions and suggestions, including a correction to the proof that multiflows have orbits. Discussions of Conley index theory with John Hahn deepened my understanding of the topic. Shannon Negaard and Cameron Theime collaborated in developing multiflow theory. Alice Nadeau encouraged me to write early and often and workshopped recalcitrant proofs with me.

The Math Climate Research Network has been a formative mathematical research community. Mary Lou Zeeman's invitation to join the MCRN Resilience Focus Group shaped my research focus and gave me a cadre of collaborator-mentors, including Sarah Iams and Alanna Hoyer-Leitzel.

Investigating resilience mathematically led not only to this thesis topic but also to interesting discussions with ecologists. I thank Allison Shaw, Lauren Sullivan, Lauren Shoemaker, Topher Weiss-Lehman, Clarence Lehman, Evelyn Strombom, Forest Isbell, Christy Portales-Reyes, Sarah Hobbie, David Tilman, and Adam Clark for giving me opportunities to learn beyond disciplinary boundaries and think carefully about the role of mathematical modeling in science.

Dedication

When we try to pick out anything by itself, we find it hitched to everything else in the universe. —John Muir, *My First Summer in the Sierra*

Of all the people, places, and things to which this thesis is tied, I particularly wish to thank...

- Bonny Fleming, who helped me navigate the PhD program and gave great advice and hugs
- Lynn Tran, who made the University of Minnesota's math library an ergonomic and pleasant space to work
- Kelsey and Kia, rockstar instructors whose RecWell strengthening classes lifted my spirits as we lifted weights
- my officemates past and present—Rosemary O'Connell, Paula Dassbach, Theodosios Douvropoulos, Lindsey Hiltner, Laurel Ohm, Cecily Santiago, Drisana Mosaphir, and Sally Jankovich—whose senses of humor and comeraderie enriched the space in 555 Vincent
- Alice Nadeau, who has been both a powerful ally through examinations, applications, and dissertations and also a true friend
- my family and friends—especially Sam Talbot, JoAnn Frekot, Carleton Meyer, Jane Meyer, Joan Kelly, Katy Noun, and Somyi Baek—whose support saw me through seeming crises and helped keep everything in perspective.

Abstract

Attractors organize long-term behavior in dynamical systems, and their robustness against perturbations reflects both the resilience of a model system and the likelihood that the inexact model captures essential features of reality. Quantifying attractor strength requires metric information beyond the topological setting of classical dynamics. We extend the theory of "intensity of attraction," first developed by McGehee for maps on compact metric spaces, to the context of continuous-time dynamics determined from a vector field on Euclidean space. Intensity measures the strength of transient dynamics within a domain of attraction, and can be computed by tracking the sets reachable from the attractor under bounded, nonautonomous control. We connect bounded control systems to multiflows, a nascent framework of multivalued dynamics. A connection between reachable sets and isolating blocks implies that an attractor's intensity not only reflects its capacity to retain solutions under time-varying perturbations, but also gives a lower bound on the distance the attractor continues in the space of vector fields.

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Chapter 1

Introduction

The motivation for this work stems from problems in resilience quantification. Popularized by ecologist C.S. Holling in the 1970s [12], the term resilience is commonly defined as a system's capacity to absorb change and disturbance while retaining its basic structure and function [31]. Resilience is not inherently good or bad; for example, a farmer might take steps to bolster the resilience of crop yields in the face of extreme climate events, while a medical practitioner might seek to overcome the resilience of a disease through treatments. In each case, managing a system with resilience in mind could help achieve normative goals.

If one must measure what is to be managed, then methods for quantifying resilience are needed. In [4], Carpenter and colleagues brought clarity to this task with their titular question "resilience of what to what?" Answering *of what* identifies system properties of interest, while answering *to what* focuses attention on relevant disturbance types. Carpenter's framework offers a guide for both empirical and theoretical explorations of resilience. Here we focus on the latter in the setting of autonomous ordinary differential equations. The equations are presumed to model some biological or physical system, and attracting states correspond to expected long-term behavior. We consider resilience *of* attractors *to* vector field perturbations—both autonomous and nonautonomous. Transient dynamics within a domain of attraction determine the effects of such perturbations, but aren't captured by common measurements of resilience based on invariant sets (for example, eigenvalues of linearization at a stable rest point or distance from attractor to basin boundary) [19]. The goal of the present work is to push beyond invariant sets to measure attractors' persistence properties in the face of vector field perturbations.

Classic dynamic concepts of stability and robustness offer qualitative rather than quantitative information about attractor persistence. For example, a structurally stable system perturbs to a topologically equivalent system for sufficiently small vector field changes in the C^1 metric [13]; precisely how small is sufficient typically remains unnamed. Similarly, isolating neighborhoods in Conley index theory are robust over a neighborhood of flows in an appropriate topology [21]. By defining robustness in terms of sufficiently close systems, one misses quantitative information about how large perturbations may become before fundamentally altering dynamic features like an attractor. But this is exactly the information needed to measure resilience. Here we expand on qualitative descriptors—stable or unstable, robust or not—to quantify *how* persistent attractors are in metric terms.

The results are organized as follows:

Chapter 2 describes our setting and presents preliminary definitions and results. In particular, we endow the state space and tangent space $\mathbb{R}^n$ with a homogeneous, translation-invariant metric and corresponding norm that measure vector field perturbations in Chapters 3, 4 and 5.

Chapter 3 formalizes vector field perturbations as bounded, nonautomous control functions. Proposition 3.6 shows that packaging all possible controlled trajectories together yields a multiflow, a nascent dynamic object studied by McGehee and students that relaxes the requirement of forward uniqueness [22, 29, 30]. Multiflows offer a framework for analyzing perturbed dynamics; in turn, bounded control systems provide a concrete example for the general theory. Theorem 3.9 departs from our control system focus to show that the set-theoretic multiflow definition does imply the existence of continuous orbits. Proposition 3.10 confirms that forming a multiflow from controlled trajectories doesn't generate any extra orbits.

Chapter 4 returns to the problem of resilience quantification. We define a quantity *intensity of attraction*, adapted from McGehee's discrete-time theory [16], that measures the control magnitude necessary to escape a domain of attraction. Intensity reflects the strength of recovery dynamics over the transient portion of a domain of attraction, and therefore differs from earlier resilience quantifiers focused solely on invariant sets. Examples 4.15 and 4.16 show how intensity generalizes the maximum recovery rate of a one-dimensional system to higher dimensional systems. Example 4.12 highlights the impact of metric choice on the value of intensity.

Chapter 5 connects an attractor's intensity to its continuation properties in the face of autonomous vector field perturbations. The key link comes from Lemma 5.11, which shows that the closures of certain reachable sets form attractor blocks with known persistence distances. In Theorem 5.14, we see that the intensity of an attractor gives a lower bound on its continuation distance in a space of autonomous vector fields. In the *resilience of what to what* framework, intensity therefore measures attractor resilience to both autonomous and nonautonomous vector field perturbations. Example 5.15 demonstrates continuation of a predator-prey limit cycle to a variety of systems within its intensity. Lastly, section 5.4 presents a proof of the upper semicontinuity of attractors based on reachable sets.

Chapter 2

Preliminaries

Consider a system of ordinary differential equations of the form

$$x' = f(x) \tag{2.1}$$

where $x \in \mathbb{R}^n$, $'$ denotes $\frac{d}{dt}$, and f maps from an open set $U \subset \mathbb{R}^n$ to $\mathbb{R}^n$.

Metrics and norms will play an important role in measuring perturbations to the vector field f. Section 2.1 presents relevant notation, definitions, and results regarding $\mathbb{R}^n$ as a metric space. Section 2.2 follows with some prerequesite background on the dynamics of equation 2.1.

2.1 $\mathbb{R}^n$ as a Metric Space

Let d be a homogeneous and translation-invariant metric on $\mathbb{R}^n$; that is: $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ satisfies

1. $d(x, y) \geq 0$ (non-negativity)
2. $d(x, y) = 0 \iff x = y$ (identity of indiscernables)
3. $d(x, y) = d(y, x)$ (symmetry)
4. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

5. $d(x,y) = d(x+a, y+a)$ (translation invariance)

6. $d(\alpha x, \alpha y) = |\alpha| d(x,y)$ (homogeneity).

Let $\|\cdot\|$ denote the corresponding norm with $\|x\| = d(x,0)$ and $d(x,y) = \|x-y\|$. Common norms include the max norm $\|(x_1, \cdots, x_n)\|_{\max} \equiv \max\{|x_i|\}$ and p-norms $\|(x_1, \cdots, x_n)\|_p \equiv (\sum x_i^p)^{1/p}$. Though we will use the Euclidean norm and metric ($p = 2$) in most examples, we intentionally develop the theory for general d and $\|\cdot\|$; particular applications would inform choices of metric and norm.

We take the vector field f in (2.1) to be globally Lipschitz on U: for any x, y in U, $\|f(x) - f(y)\| \leq L \|x - y\|$ for some constant L that depends on the choice of norm. We also assume that f belongs to the space of bounded functions from U to $\mathbb{R}^n$, with finite norm $\|f\|_{\sup} \equiv \sup_{x \in U} \|f(x)\|$.

The following definitions and notations are standard in this metric setting. Let

$$\mathrm{dist}(X, Y) \equiv \inf\{d(x,y) \mid x \in X, y \in Y\}$$

denote the distance between two sets X, $Y \subset \mathbb{R}^n$. (A single point set $\{x\}$ will not be distinguished from its element x.) For $\epsilon > 0$ and $x \in \mathbb{R}^n$, the ϵ ball about x will be written

$$B_\epsilon(x) \equiv \{y \in \mathbb{R}^n \mid d(x,y) < \epsilon\}.$$

More generally, we denote the ϵ-neighborhood of a set $S \subset \mathbb{R}^n$ as

$$\mathcal{N}_\epsilon(S) \equiv \{x \in \mathbb{R}^n \mid d(x,S) < \epsilon\}.$$

The metric topology is assumed throughout. For $S \subset \mathbb{R}^n$, $\mathrm{int}(S)$ and $\overline{S}$ denote the interior and closure of S, respectively. A set N is a neighborhood of S if $S \subset \mathrm{int}(N)$. The following lemma states that a compact set has arbitrarily close compact neighborhoods; its straightforward proof is omitted.

Lemma 2.1. *If $S \subset \mathbb{R}^n$ is compact, then for any neighborhood N of S there exists a compact set $K \subset int(N)$ that is also a neighborhood of S.*

2.2 Continuous-Time Dynamics

We return to the dynamics of equation 2.1. Because f is Lipschitz continuous, there exists a unique local solution $x(t)$ to the initial value problem $x' = f(x)$, $x(0) = x_0$. These solutions generate a local flow.

Definition 2.2. For $D \subset \mathbb{R} \times U$, a *local flow* associated with equation 2.1 is a function $\varphi : D \to U$ given by $\varphi(t, x_0) = x(t)$, where $x(t)$ solves the initial value problem $x' = f(x)$, $x(0) = x_0$.

Remark 2.3. Trajectories leaving U pose the barrier to defining a global flow $\varphi : \mathbb{R} \times U$ from globally Lipschitz f. For simplicity we will assume that φ is well-defined on any time domain of interest.

Fixing the time coordinate of φ yields a time-t map $\varphi^t : U \to U$, given by $x \mapsto \varphi(t, x)$. Functions such as the flow and the time-t map will be extended in the natural way to take sets as arguments. For example, if $T \subset \mathbb{R}$ and $S \subset \mathbb{R}^n$, then $\varphi(T, S) \equiv \bigcup_{t \in T} \bigcup_{x \in S} \varphi(t, x)$ and $\varphi^t(S) \equiv \bigcup_{x \in S} \varphi^t(x)$.

Definitions 2.4 and 2.5 build toward the definition of an attractor (Defintion 2.7).

Definition 2.4. A set $S \subset U$ is *forward invariant* under the flow φ if $\varphi^t(S) \subset S$ for all $t > 0$. S is *invariant* if $\varphi^t(S) \subset S$ for all t.

The long-term behavior of invariant sets is clear: they stay put. The ultimate fate of a set in general is captured by its omega limit set.

Definition 2.5. The omega limit set of a set $S \subset U$ is $\omega(S) \equiv \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} \varphi^t(S)}$.

Taking omega limit sets preserves inclusion; the proof follows immediately from Definition 2.5.

Lemma 2.6. $S_1 \subset S_2 \implies \omega(S_1) \subset \omega(S_2)$

The pieces are now in place to define an attractor for a flow.

Definition 2.7. An attractor is a compact, nonempty invariant set A such that $A = \omega(N)$ for some neighborhood N of A.

The statement $A = \omega(N)$ indicates that the neighborhood N is "attracted" asymptotically to A in forward time. The collection of all points that tend to A in forward time is called A's domain of attraction.

Definition 2.8. The domain of attraction of an attractor A is $\mathcal{D}(A) = \{x \in U \mid \emptyset \neq \omega(x) \subset A\}$.

This concludes our brief review of continuous-time dynamics. The interested reader can find more detailed discussion and examples of flows and attractors in [11] (introductory) and [7] (advanced).

Chapter 3

Control Systems as Multiflows

McGehee defined intensity of attraction for discrete dynamical systems in terms of the smallest errors sufficient to cause a perturbed trajectory to escape from an attractor's domain[16]. Given a continuous map ϕ on a metric space, McGehee formalized perturbed trajectories as ϵ-pseudo-orbits: sequences $\{x_n\}$ such that $d(x_i, \phi(x_{i-1})) < \epsilon$ for fixed error bound $\epsilon > 0$. The study of ϵ-pseudo-orbits in a metric space X connects naturally to the study of relations, subsets of $X \times X$ that can also be interpreted as set-valued maps [17]. In particular, the relation ϕ_ϵ that sends x to a ball of radius ϵ centered on $\phi(x)$ gives a dynamical system without forward uniqueness that allows tracking of many ϵ-pseudo-orbits at once.

In the context of the continuous-time dynamical system given by equation (2.1), bounded non-autonomous control functions offer an analog to discrete ϵ-errors. Simultaneously considering an entire collection of such control functions yields a continuous-time dynamical system lacking forward uniqueness. This continous time analog to the discrete relation, termed a "multiflow" by McGehee, connects naturally to our development of a continuous theory of intensity of attraction. In this chapter we introduce the setting of bounded control functions (Section 3.1) and prove that a collection of bounded control functions gives a multiflow (Section 3.2). We show that in general, the set-based definition of multiflows implies the existence of orbits (Section 3.3). However, any orbits of the particular multiflow built from controlled trajectories are themselves controlled trajectories (Section 3.4).

3.1 Bounded Time-Dependent Control

We introduce time-dependent perturbations to the system (2.1) via locally integrable and essentially bounded control functions $g : \mathbb{R} \supset I \to \mathbb{R}^n$, yielding a perturbed system

$$\frac{dx}{dt} = f(x) + g(t). \tag{3.1}$$

Note that g can represent unknown and unintentional perturbations to a dynamical system, in addition to the planned interventions associated with the standard "control function" nomenclature. For example, g could represent climate variation in an ecological model, rather than direct human control.

The precise bound on g plays an important role in Chapters 4 and 5 and depends on the norm being used. As the functions f and g both map to $\mathbb{R}^n$, we employ the same norm $\|\cdot\|$ on each range. Given a time interval $I \subset \mathbb{R}$, we concern ourselves with functions g in the space $L^\infty(I, \mathbb{R}^n)$ equipped with the norm $\|g\|_\infty \equiv \inf\{c \geq 0 : \mu(\{x \in I : \|g(x)\| > c\}) = 0\}$.

With $\|g\|_\infty$ finite, the right hand side of (3.1) satisfies the the Carathéodory conditions on $U \times I$: the function $F(x,t) = f(x) + g(t)$ is measurable in t for fixed x, continuous in x for fixed t, and (without loss of generality) bounded on any compact subset of $I \times U$ by the constant and hence integrable function $\|f\|_{\sup} + \|g\|_\infty$. Furthermore, it has the Lipschitz property in x that $\|F(x,t) - F(y,t)\| = \|f(x) - f(y)\| \leq L\,\|x - y\|$. It follows (see, e.g. [5, 10]) that for any initial condition $x(t_0) = x_0$ there exists a unique local solution $x(t)$ to (3.1), in the extended sense that

$$x(t) = x(t_0) + \int_{t_0}^{t} \left(f(x(s)) + g(s)\right) ds \tag{3.2}$$

$$\text{and} \qquad x'(t) = f(x) + g(t) \text{ almost everywhere.} \tag{3.3}$$

The characterization (3.2) of solutions provides an analytic tool, while equation 3.3 highlights the meaning of $\|g\|_\infty$ as a maximum perturbation (or control) rate. Unless otherwise specified, we will assume that a unique local solution $x(t)$ to (3.1) can be continued on all of any relevant time interval I.

In general, solutions to Carathéodory differential equations are absolutely continuous. In the specific class of systems (3.1), solutions have an even stronger property:

Lemma 3.1. *Solutions to equation 3.1 are Lipschitz continuous with Lipschitz constant* $\|f\|_{sup} + \|g\|_\infty$.

Proof:

$$\begin{aligned}
\|x(s) - x(t)\| &= \left\| \int_t^s (f(x(\tau)) + g(\tau))\, d\tau \right\| && \text{property (3.2) of solutions} \\
&\leq \int_t^s \|f(x(\tau)) + g(\tau)\|\, d\tau \\
&\leq \int_t^s (\|f(x(\tau))\| + \|g(\tau)\|)\, d\tau && \Delta\text{ineq., monotonicity} \\
&\leq \int_t^s \|f(x(\tau))\|\, d\tau + \int_t^s \|g(\tau)\|\, d\tau && \text{linearity} \\
&\leq \|f\|_{\text{sup}} |s-t| + \|g\|_\infty |s-t| \\
&= (\|f\|_{\text{sup}} + \|g\|_\infty)|s-t|. && \square
\end{aligned}$$

Existence and uniqueness of solutions to (3.1) permit extension of standard flow notation to include the control function g, in a manner similar to [6]:

Definition 3.2. For a given f and time interval $[0, T]$, define the controlled flow $\varphi(\,\cdot\,, \,\cdot\,, \cdot\,; f) : [0, T] \times U \times L^\infty([0, T], \mathbb{R}^n) \to \mathbb{R}^n$ by $\varphi(t, x_0, g; f) = x(t)$, where $x(s)$ is the solution to $x' = f(x) + g(s)$, $x(0) = x_0$. For $S \subset \mathbb{R}^n$, let $\varphi(t, S, g; f) = \bigcup_{x_0 \in S} \varphi(t, x_0, g; f)$.

We will omit the vector field f when it is clear from context. To emphasize depedence on initial conditions, we will write $\varphi(t, \cdot, g) = \varphi_g^t(\cdot)$. In what follows, we move fluidly between describing solutions to (3.1) with the notation $x(t)$ and with the flow notation of Definition 3.2. The latter has the advantage of making g visible.

Continuity of $\varphi(t, x, g)$ with respect to t and x is well-known (see [11]). Lemma 3.3 establishes continuity of $\varphi(t, x, g)$ with respect to control input g.

Lemma 3.3. *Fix a time interval* $[0, T]$*. For any* ϵ*, there exists a* δ *such that* $\|g - h\|_\infty < \delta$ *implies* $\|\varphi(t, x, g) - \varphi(t, x, h)\| < \epsilon$ *for all* $t \in [0, T]$.

Proof: We apply Gronwall's Inequality to the non-negative, continuous function $u(t) \equiv \|\varphi(t,x,g) - \varphi(t,x,h)\|$. Fixing ϵ, equation (3.2) gives that for any $x \in U$ and any $t \in [0,T]$ that

$$\begin{aligned}
& \varphi(t,x,g) = x + \int_0^t \big[f(\varphi(s,x,g)) + g(s)\big]ds \\
\text{and} \qquad & \varphi(t,x,h) = x + \int_0^t \big[f(\varphi(s,x,h)) + h(s)\big]\ ds \\
\text{so} \qquad & u(t) = \left\| \int_0^t \big[f(\varphi(s,x,g)) + g(s) - f(\varphi(s,x,h)) - h(s)\big]ds \right\| \\
& \leq \int_0^t \|f(\varphi(s,x,g)) - f(\varphi(s,x,h))\|\, ds + \int_0^t \|g(s) - h(s)\|\, ds \\
& \leq \int_0^t L\, \|\varphi(s,x,g) - \varphi(s,x,h)\|\, ds + t\|g-h\|_\infty \\
& = \int_0^t Lu(s)ds + t\|g-h\|_\infty \\
& \leq T\|g-h\|_\infty + L\int_0^t u(s)ds
\end{aligned}$$

Because there exist constants $C_1 = T\|g-h\|_\infty$ and $C_2 = L$ such that the non-negative, continuous function u satisfies $u(t) \leq C_1 + C_2\int_0^t u(s)ds$ for all $t \in [0,T]$, Gronwall's Inequality implies that $u(t) \leq C_1 e^{C_2 t}$ for all $t \in [0,T]$. Coupled with $t \leq T$ and the definition of $u(t)$, this gives

$$\|\varphi(t,x,g) - \varphi(t,x,h)\| \leq T\|g-h\|_\infty e^{LT}.$$

Choosing $\delta = \frac{\epsilon}{Te^{LT}}$, we have that $\|g-h\|_\infty < \delta$ implies $\|\varphi(t,x,g) - \varphi(t,x,h)\| < \epsilon$ for all $t \in [0,T]$ and all $x \in \mathbb{R}^n$, as desired. $\square$

The controlled flow of Definition 3.2 represents outcomes under a single function $g \in L^\infty(I,\mathbb{R}^n)$, in alignment with g representing an intentional human intervention in a controlled system. However, when g represents an uncertain disturbance, it is natural to consider trajectories corresponding to an entire family of such functions. We pursue this viewpoint further in section 3.2.

3.2 Bounded Control Systems Give Multiflows

For discrete-time dynamics, relations generalize maps by relaxing the requirement of forward uniqueness. For continuous time, McGehee proposes the corresponding generalization of a flow:

Definition 3.4. For X a compact metric space, a closed subset $\Phi \subset X \times X \times [0,\infty)$ is a *multiflow* on X if

1. $\Phi^0 = \mathrm{id}$ *(identity property)*
2. $\Phi^s \circ \Phi^t = \Phi^{s+t}$ for all $t \geq 0$ and $s \geq 0$ *(semigroup homomorphism property)*,

where $\Phi^t = \{(a,b) : (a,b,t) \in \Phi\}$, $\Phi^s \circ \Phi^t = \{(a,c) \mid \exists b \in X \text{ s.t. } (a,b) \in \Phi^t,\ (b,c) \in \Phi^s\}$.

The two defining properties of a multiflow resemble those of a flow, but fixing the time coordinate in a multiflow yields a time-t *relation* Φ^t rather than a time-t map φ^t. We treat such a relation $R \subset X \times X$ as a set-valued map with $R(x)$ given by $\{y \in X \mid (x,y) \in R\}$. For $S \subset X$, we write $R(S)$ to mean $\bigcup_{x \in S} R(x)$.

The multiflow definition is currently under development as a platform for studying nonsmooth dynamics that arise in physical applications [29, 22]. In our setting, multiflows offer a way to simultaneously consider all systems of form (3.1) with a fixed vector field f and bound r on $\|g\|_\infty$. To streamline our discussion of bounded families of control functions, we write $B_r(0) = \{g \in L^\infty(I, \mathbb{R}^n) \mid \|g\|_\infty < r\}$ and $\overline{B_r(0)} = \{g \in L^\infty(I, \mathbb{R}^n) \mid \|g\|_\infty \leq r\}$ to describe the collections of control functions strictly bounded below r and bounded by r, respectively.

Definition 3.5. Suppose the dynamics of interest for equation 3.1 take place in a compact set $X \subset \mathbb{R}^n$. Then $\Phi_{r;f} \subset X \times X \times [0,\infty)$ will denote the set

$$\{(a,b,T) \mid \text{for some } g \in \overline{B_r(0)} \subset L^\infty([0,T], \mathbb{R}^n), \text{ the solution } x : [0,T] \to \mathbb{R}^n \\ \text{to } x' = f(x) + g(t) \text{ satisfies } x(0) = a, x(T) = b\}.$$

In other words, (a,b,T) is in $\Phi_{r;f}$ if there exists an integrable control / disturbance function essentially bounded by r that can move the state of the system $x' = f(x) + g(t)$

from a to b in time T. When the vector field f is clear from context or general, we will drop it from the subscript, and use $\Phi_r \equiv \Phi_{r;f}$.

Proposition 3.6. *Φ_r is a multiflow.*

Proof: The identity property $\Phi_r^0 = id$ follows immediately from Definition 3.5, since a controlled / disturbed system (3.1) can move from a to b in time 0 if and only if $a = b$.

For the semigroup property, first suppose that $(a, c) \in \Phi_r^s \circ \Phi_r^t$. This implies the existence of a point $b \in X$ such that $(a, b) \in \Phi_r^t$ and $(b, c) \in \Phi_r^s$. There must be functions $g_1 \in L^\infty([0, t], \mathbb{R}^n), g_2 \in L^\infty([0, s], \mathbb{R}^n)$, each essentially bounded by r, such that $\varphi_{g_1}^t(a) = b$ and $\varphi_{g_2}^s(b) = c$. Construct $\hat{g} : [0, s + t] \to \mathbb{R}^n$ as

$$\hat{g}(\tau) = \begin{cases} g_1(\tau) & \tau \leq t \\ g_2(\tau - t) & \tau > t. \end{cases} \tag{3.4}$$

This spliced-together disturbance function $\hat{g}$ is again locally integrable and essentially bounded by r. By construction it yields a solution curve such that $\varphi_{\hat{g}}^{s+t}(a) = c$. Therefore $(a, c) \in \varphi_r^{s+t}$ and the inclusion $\Phi_r^s \circ \Phi_r^t \subset \Phi_r^{s+t}$ is established.

On the other hand, suppose $(a, c) \in \Phi_r^{s+t}$. Then there exists a locally integrable disturbance $g \in \overline{B_r(0)} \subset L^\infty([s + t], \mathbb{R}^n)$ such that $\varphi_g^{s+t}(a) = c$. Let $\varphi_g^t(a) = b$. Define $g_1 = g\big|_{[0,t]}$ and $g_2 : [0, s] \to \mathbb{R}^n$ by $g_2(\tau) = g(\tau + t)$. One can easily confirm that $g_1 \in L^\infty([0, t], \mathbb{R}^n)$ and $g_2 \in L^\infty([0, s], \mathbb{R}^n)$. We have that $\varphi_{g_1}^t(a) = b$ and $\varphi_{g_2}^s(b) = c$, so $(a, b) \in \Phi_r^t$ and $(b, c) \in \Phi_r^s$. Therefore $(a, c) \in \Phi_r^s \circ \Phi_r^t$, $\Phi_r^{s+t} \subset \Phi_r^s \circ \Phi_r^t$, and the semigroup property is established.

It remains to be shown that Φ_r is closed. Let (a, b, T) be a limit point of Φ_r. Then there exists a sequence of disturbances $\{\tilde{g}_k : [0, T_k] \to \mathbb{R}^n\}$, with $\tilde{g}_k \in \overline{B_r(0)} \subset L^\infty([0, T_k], \mathbb{R}^n)$, such that corresponding solutions $\{\tilde{x}_k : [0, T_k] \to X\}$ of $dx/dt = f(x) + g_k(t)$ converge at endpoints $\tilde{x}_k(0) \to a$ and $\tilde{x}_k(T_k) \to b$, while simultaneously $T_k \to T$.

To work with a family of solutions on a common domain, we modify the $\tilde{g}_k$ as follows.

If $T_k < T$ then let

$$g_k(t) = \begin{cases} \tilde{g}_k(t) & 0 < t \leq T_k \\ 0 & T_k < t \leq T. \end{cases} \tag{3.5}$$

If $T_k \geq T$ then let

$$g_k = \tilde{g}_k\big|_{[0,T]}. \tag{3.6}$$

Note that the g_k are still locally integrable and essentially bounded by r, so $\{g_k\} \subset \overline{B_r(0)} \subset L^\infty([0,T], \mathbb{R}^n)$.

We define a corresponding sequence of solutions $\{x_k : [0,T] \to X\}$ to $dx/dt = f(x) + g_k(t)$. The integral characterization of solutions (3.2) gives that

$$x_k(t) = x_k(0) + \int_0^t (f(x_k(s)) + g_k(s))ds. \tag{3.7}$$

Slight modification of the proof of Lemma 3.1 yields that the family $\{x_k\}$ shares the Lipschitz constant $\|f\|_{\text{sup}} + r$. We next apply Grönwall's inequality to show that $\{x_k\}$ is uniformly bounded. Let $u_k : [0,T] \to \mathbb{R}$ be given by $u_k(t) = \|x_k(t) - x_k(0)\|$. Clearly u_k is non-negative and continuous in t for all k. Futher, there exist constants $C_1 = T(\|f\|_{\text{sup}} + r)$ and $C_2 = L$ such that $u_k(t) \leq C_1 + \int_0^t C_2 u_k(s)ds$ for all k and all $t \in [0,T]$:

$$\begin{aligned} u_k(t) &= \left\| \int_0^t (f(x_k(s)) + g_k(s))\, ds \right\| \\ &\leq \int_0^t \big\|f(x_k(s))\big\|ds + \int_0^t \big\|g_k(s)\big\|ds \\ &\leq \int_0^t \big\|f(x_k(s)) - f(x_k(0))\big\|ds + \int_0^t \big\|f(x_k(0))\big\|ds + \int_0^t \big\|g_k(s)\big\|ds \\ &\leq \int_0^t L\big\|x_k(s) - x_k(0)\big\|ds + t(\|f\|_{\text{sup}} + r) \\ &\leq T(\|f\|_{\text{sup}} + r) + \int_0^t Lu_k(s)ds \end{aligned}$$

Grönwall's inequality thus implies that $0 \leq u_k(t) \leq C_1 e^{C_2 t} \leq C_1 e^{C_2 T}$ for all $t \in [0,T]$.

This uniform bound on $\{u_k\}$ in turn gives a uniform bound on $\{x_k\}$:

$$\|x_k(t)\| = \|x_k(t) - x_k(0) + x_k(0)\| \leq u_k(t) + \|x_k(0)\| \leq C_1 e^{C_2 T} + \max_{x \in N} \|x\|$$

where N is a compact neighborhood of a containing (without loss of generality) all $x_k(0)$.

Because $\{x_k\}$ is uniformly bounded and uniformly Lipschitz, by the Arzela-Ascoli Theorem a subsequence $\{x_j\} \subset \{x_k\}$ converges uniformly to a function $x_* : [0,T] \to X$ with the same Lipschitz constant $\|f\|_{\sup} + r$.

Closedness of Φ_r will follow once we show that $x_*(0) = a$, $x_*(T) = b$, and that there exists a $g_* \in \overline{B_r(0)} \subset \mathcal{G}[0,T]$ for which $x_*(t)$ is the solution to $x' = f(x) + g_*(t)$.

The first fact follows immediately from construction of $\{x_j\}$, as $x_*(0) = \lim_{j\to\infty} x_j(0) = \lim_{k\to\infty} x_k(0) = a$. Towards the second, we use the ϵ, δ definition of a limit to show that $\lim_{j\to\infty} x_j(T) = b$. Fix $\epsilon > 0$. Since $T_j \to T$, there exists $J_1 \in \mathbb{N}$ so that $|T_j - T| < \frac{\epsilon}{2(\|f\|_{\sup}+r)}$ whenever $j > J_1$. Further, since $\tilde{x}_j(T_j) \to b$, there exists $J_2 \in \mathbb{N}$ so that $\|\tilde{x}_j(T_j) - b\| < \epsilon/2$ whenever $j > J_2$. Let $J = \max\{J_1, J_2\}$. Then for any $j > J$ such that $T < T_j$ (giving $x_j(T) = \tilde{x}_j(T)$), we have

$$\begin{aligned}
\|x_j(T) - b\| &\leq \|\tilde{x}_j(T) - \tilde{x}_j(T_j)\| + \|\tilde{x}_j(T_j) - b\| \\
&= \left\| \int_{T_j}^{T} [f(\tilde{x}_j(s)) + \tilde{g}_j(s)]\, ds \right\| + \|\tilde{x}_j(T_j) - b\| \\
&\leq |T_j - T|(\|f\|_{\sup} + r) + \|\tilde{x}_j(T_j) - b\| \\
&< \epsilon
\end{aligned}$$

For any $j > J$ such that $T_j < T$ (giving $x_j(T_j) = \tilde{x}_j(T_j)$), a similar argument yields $\|x_j(T) - b\| \leq \|x_j(T) - x_j(T_j)\| + \|\tilde{x}_j(T_j) - b\| < \epsilon$, as desired. Hence $x_*(T) = \lim_{j\to\infty} x_j(T) = b$.

Towards the final fact, note that for each x_j, equation (3.7) implies

$$x_j(t) - x_j(0) - \int_0^t f(x_j(s))ds = \int_0^t g_j(s)ds. \tag{3.8}$$

Clearly $x_j(t) - x_j(0)$ converges uniformly to $x_*(t) - x_*(0)$ as $j \to \infty$. A straightforward argument based on uniform convergence of $\{x_j\}$, Lipschitz continuity of f, and boundedness of the time interval $[0, T]$ establishes that $\int_0^t f(x_j(s))ds$ converges uniformly to $\int_0^t f(x_*(s))ds$ as $j \to \infty$. Therefore, the right-hand side $\int_0^t g_j(s)ds \equiv G_j(t)$ converges uniformly to the function

$$G_*(t) = x_*(t) - x_*(0) - \int_0^t f(x_*(s))ds. \tag{3.9}$$

Boundedness of each g_j by r implies that each G_j is Lipschitz continuous with Lipschitz constant r. As the uniform limit of uniformly Lipschitz functions, G_* shares the same Lipschitz constant r.

Let

$$g_*(t) = \frac{d}{dt}G_*(t) = \frac{d}{dt}x_*(t) - f(x_*(t)) \tag{3.10}$$

where the derivatives denote integrable functions on $[0, T]$ that agree with the derivatives of the corresponding absolutely continuous functions almost everywhere. Note that the derivative g_* of G_* is essentially bounded by the latter's Lipschitz constant r.

We have demonstrated a function g_* that is locally integrable and essentially bounded by r. Further, the trajectory x_* with $x_*(0) = a$ and $x_*(T) = b$ satisfies $x_*' = f(x_*) + g_*(t)$. By definition 3.5, $(a, b, T) \in \Phi_r$. The set Φ_r therefore contains its limit points, and is closed. $\square$

3.3 Multiflows Have Orbits

We use the term "orbit" to refer to a path that obeys the dynamic rules encoded in a multiflow:

Definition 3.7. An *orbit* of a multiflow Φ is a continuous function ψ from an interval $J \subset \mathbb{R}$ to state space X (here, a compact subset of $\mathbb{R}^n$), such that $(\psi(t), \psi(s), s-t) \in \Phi$ whenever $s, t \in J$ and $s > t$.

It seems natural that the multiflow Φ_r, which is defined in terms of solution trajectories, has orbits. However, for a general multiflow whose only known properties are given by Definition 3.4, existence of orbits is not immediately clear. McGehee posed the following question:

Question 3.8. Let Φ be a multiflow on the compact metric space X, and let $(p, q) \in \Phi^s$. Is there an orbit $\psi : [0, s] \to X$ satisfying $\psi(0) = p$ and $\psi(s) = q$?

Theorem 3.9. *The answer to Question 3.8 is "yes."*

Proof: The proof consists of four steps:

1. Construct a function from a dense subset of $[0, s]$ to X using the semigroup property of Φ.

2. Demonstrate Cauchy-continuity of the function.

3. Extend the function in the natural way to a continuous function on all of $[0, s]$.

4. Show that the extension satisfies the required conditions for an orbit of Φ.

For step 1, consider the dense subset of $[0, s]$ formed by scaling the dyadic rationals on $[0, 1]$ by s:

$$D = \{rs : r = \frac{a}{2^b} \text{ for some } \; b \in \mathbb{N},\; a \in [0, 2^b] \cap \mathbb{Z}\}.$$

We define a function $\psi : D \to X$ recursively. Let D_j denote the finite subset $\left\{\frac{i}{2^j} \cdot s\right\}_{i=0}^{i=2^j} \subset D$. Define ψ on D_0 by chosing $\psi(0) = p$ and $\psi(s) = q$. Note that $(p, q, s) \in \Phi$ by assumption. For $j \geq 0$, extend the domain of ψ from D_j to D_{j+1} as follows. If $t \in D_{j+1} \backslash D_j$ then the neighboring dyadics in D_j are $t_- = t - \frac{s}{2^{j+1}}$ and $t_+ = t + \frac{s}{2^{j+1}}$. We assume that

$$(\psi(t_-), \psi(t_+), \frac{s}{2^j}) \in \Phi, \tag{3.11}$$

as is true for $j = 0$. (Justification of (3.11) for $j > 0$ will come momentarily.) Then the semigroup property (Definition 3.4) implies that there exists at least one point $\xi \in X$

with $(\psi(t_-), \xi, \frac{s}{2^{j+1}}) \in \Phi$ and $(\xi, \psi(t_+), \frac{s}{2^{j+1}}) \in \Phi$. Choose one such point and define $\psi(t) = \xi$. Doing this 2^j times defines ψ_{j+1} on each element of $D_{j+1} \backslash D_j$. By construction, $(\psi(t), \psi(t + \frac{s}{2^{j+1}}), \frac{s}{2^{j+1}}) \in \Phi$ for each $t \in D_{j+1} \backslash \{s\}$, so (3.11) continues to hold for $j > 0$. The semigroup property implies that $(\psi_j(t_1), \psi_j(t_2), t_2 - t_1) \in \Phi$ for any $t_1, t_2 \in D_j$ with $t_2 \geq t_1$, not just adjacent dyadics. Finite recursive steps define $\psi(t)$for any $t \in D$, completing step one.

In step two, we demonstrate that $\psi : D \to X$ is Cauchy-continuous. Let $\{t_i\}$ be a Cauchy sequence in D. Assume for the sake of contradiction that its image $\{\psi(t_i)\}$ in X is *not* a Cauchy sequence. Then there exists an $\epsilon > 0$ such that for any N, $m, n > N$ does not imply $d(\psi(t_m), \psi(t_n)) < \epsilon$. Consider a sequence of pairs $\{(\psi(t_{m_N}), \psi(t_{n_N}))\}_{N=1}^{\infty}$ such that $m_N, n_N > N$ and $d(\psi(t_{m_N}), \psi(t_{n_N})) \geq \epsilon$. Compactness of $X \times X$ with respect to the product metric d_∞[1] implies there exists a subsequence

$$\left\{ \left(\psi(t_{m_k}), \psi(t_{n_k}) \right) \right\}_{k=1}^{\infty} \tag{3.12}$$

that converges to $(L_1, L_2) \in X \times X$ in d_∞. It follows that $d(L_1, L_2) \geq \epsilon$, so $L_1 \neq L_2$. Without loss of generality[2] suppose $t_{n_k} > t_{m_k}$. Since for each k the time points t_{n_k} and t_{m_k} both belong to some finite $D_j \subset D$, there exists a corresponding sequence of multiflow elements

$$\left\{ \left(\psi(t_{m_k}), \psi(t_{n_k}), t_{n_k} - t_{m_k} \right) \right\}_{k=1}^{\infty} \subset \Phi. \tag{3.13}$$

This sequence limits to $(L_1, L_2, 0) \in X \times X \times \mathbb{R}$, with the zero in the last component following from the Cauchy property of the sequence $\{t_i\}$ along with the construction of $\{t_{n_k}\}$ and $\{t_{m_k}\}$. The closed multiflow Φ must contain its limit point $(L_1, L_2, 0)$. But $L_1 \neq L_2$, contradicting the identity property of Φ (Definition 3.4). We conclude that $\{\psi(t_i)\}$ *is* Cauchy-continuous.

In step three, we extend the function $\psi : D \to X$ uniquely to a continuous function

[1] $d_\infty((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), d(y_1, y_2)\}$

[2] Taking further subsequences can ensure consistent ordering throughout the sequence; the case $t_{n_k} < t_{m_k}$ follows similarly.

$\hat{\psi} : [0, s] \to X$ by

$$\hat{\psi}(t) = \begin{cases} \psi(t) & \text{if } t \in D \\ \lim_{t_k \to t} \psi(t_k) & \text{if } t \in [0, s] \backslash D \end{cases}. \tag{3.14}$$

where $\{t_k\}$ is any sequence in D that limits to $t \in [0, s]\backslash D$. Such unique extension is possible because D is dense in $[0, s]$, ψ is Cauchy-continuous, and X, a compact metric space, is complete.

In step four, we confirm that $\hat{\psi}$ is the desired orbit for Φ. Clearly $\hat{\psi}$ is a continuous function from $[0, s]$ to X satisfying $\hat{\psi}(0) = p$ and $\hat{\psi}(s) = q$. It remains to confirm that $(\psi(t_1), \psi(t_2), t_2 - t_1) \in \Phi$ whenever $t_1 \in [0, s]$, $t_2 \in [0, s]$, and $t_2 \geq t_1$. The case in which both t_1 and t_2 are elements of D was demonstrated in step (1). If either t_1 or t_2 is not an element of D, consider sequences $D \supset \{t_i\} \to t_1$ and $D \supset \{t_j\} \to t_2$. The collection of multiflow elements $\{(\psi(t_i), \psi(t_j), t_j - t_i)\}$ has $(\psi(t_1), \psi(t_2), t_2 - t_1)$ as a limit point, which must belong to closed Φ. This completes the proof. $\square$

3.4 Orbits of Control Multiflows Are Solutions

The fact that the definition of multiflow guarantees orbits (Theorem 3.9) raises the question of whether the multiflow Φ_r, defined in terms of controlled trajectories (Definition 3.5), could have orbits *besides* the trajectories from which it is built. Proposition 3.10 says no.

Proposition 3.10. *Let J be an interval in $\mathbb{R}$. Any orbit $\psi : J \to X$ of Φ_r satisfies $\psi'(t) = f(\psi(t)) + h(t)$ for some $h \in \overline{B_r(0)} \subset L^\infty(J, \mathbb{R}^n)$, and hence is a solution of the bounded control equation (3.1) from which Φ_r is built.*

Proof: We first show that any orbit ψ of Φ_r is Lipschitz continuous. For $s, t \in J$ with $s > t$ we have that $(\psi(t), \psi(s), s - t) \in \Phi_r$, implying $\|\psi(s) - \psi(t)\| = \|x(s - t) - x(0)\|$ for a solution $x(t)$ to $x' = f(x) + g(t)$ with $g \in \overline{B_r(0)} \subset L^\infty([0, s - t], \mathbb{R}^n)$. Lemma 3.1 gives that $\|x(s - t) - x(0)\| \leq (\|f\|_{\text{sup}} + r)|s - t|$, so ψ shares the Lipschitz constant of controlled solutions.

Lipschitz continuity of ψ implies absolute continuity, so ψ is almost everywhere differentiable. Let $\overline{\psi'} : J \to \mathbb{R}^n$ denote a locally integrable function that agrees with ψ' where ψ' is defined. Define the locally integrable function $h : J \to \mathbb{R}^n$ by $h(t) = \overline{\psi'}(t) - f(\psi(t))$. To confirm that h is essentially bounded by r, fix t in the interior of J such that $\psi'(t)$ exists. Then

$$\begin{aligned}
\|h(t)\| &= \left\|\psi'(t) - f(\psi(t))\right\| \\
&= \left\|\lim_{\delta\to 0} \frac{\psi(t+\delta) - \psi(t)}{\delta} - f(\psi(t))\right\| \\
&= \lim_{\delta\to 0} \left\|\frac{\psi(t+\delta) - \psi(t) - \delta f(\psi(t))}{\delta}\right\|
\end{aligned}$$

We next confirm that for any $\epsilon > 0$, the inequality $\left\|\frac{\psi(t+\delta) - \psi(t) - \delta f(\psi(t))}{\delta}\right\| \leq r + \epsilon$ holds for all sufficiently small $\delta > 0$. Fix $\epsilon > 0$. Definition 3.7 gives that $(\psi(t), \psi(t+\delta), \delta) \in \Phi_r$ for any $\delta > 0$ such that $t + \delta \in J$. So by definition 3.5, there exists a control function $g_\delta \in \overline{B_r(0)} \subset L^\infty([0,\delta], \mathbb{R}^n)$ with solution x_δ to $x' = f(x) + g_\delta(t)$ satisfying $x_\delta(0) = \psi(t)$ and $x_\delta(\delta) = \psi(t+\delta)$. Hence

$$\begin{aligned}
\left\|\frac{\psi(t+\delta) - \psi(t) - \delta f(\psi(t))}{\delta}\right\| &= \left\|\frac{x_\delta(\delta) - x_\delta(0) - \delta f(x_\delta(0))}{\delta}\right\| \\
&= \left\|\frac{\displaystyle\int_0^\delta \big[f(x_\delta(\tau)) + g_\delta(\tau)\big]\,d\tau - \delta f(x_\delta(0))}{\delta}\right\| \\
&= \left\|\frac{\displaystyle\int_0^\delta \big[f(x_\delta(\tau)) - f(x_\delta(0)) + g_\delta(\tau)\big]\,d\tau}{\delta}\right\| \\
&\leq \frac{1}{\delta}\int_0^\delta \|f(x_\delta(\tau)) - f(x_\delta(0))\|\,d\tau + \frac{1}{\delta}\int_0^\delta \|g_\delta(\tau)\|\,d\tau \\
&\leq \frac{1}{\delta}\int_0^\delta L\,\|x_\delta(\tau) - x_\delta(0)\|\,d\tau + r
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\delta}\int_0^{\delta} L(\|f\|_{\sup}+r)|\tau| d\tau + r \qquad\qquad \text{(Lemma 3.1)}\\
&= \frac{L(\|f\|_{\sup}+r)}{\delta}\left[\frac{\tau^2}{2}\right]_0^{\delta} + r\\
&= \frac{\delta L(\|f\|_{\sup}+r)}{2} + r\\
&\le r+\epsilon \qquad\qquad \text{for all } \delta \in \left(0, \frac{2\epsilon}{L(\|f\|_{\sup}+r)}\right)
\end{aligned}
$$

A similar argument shows that $\left\|\dfrac{\psi(t+\delta)-\psi(t)-\delta f(\psi(t))}{\delta}\right\| \le r+\epsilon$ for all $\delta \in \left(\frac{-2\epsilon}{L(\|f\|_{\sup}+r)}, 0\right)$. Because $\|h(t)\|$ is bounded by $r+\epsilon$ for any $\epsilon > 0$, it is bounded by r. This bound holds on all of J except for possibly a set of measure zero. By construction, $\psi'(t) = f(\psi(t))+h(t)$, and we have confirmed that $h \in \overline{B_r(0)} \subset L^{\infty}(J, \mathbb{R}^n)$. Thus any orbit ψ of Φ_r satisfies the bounded control equation $x' = f(x)+g(t)$, $\|g\|_{\infty} \le r$. $\square$

Chapter 4

Intensity of Attraction

Gaps in the resilience quantification literature [19] provide one motivation for studying intensity of attraction. Ecological modelers commonly quantify resilience—roughly, a system's capacity to absorb change and disturbance while maintaining its basic structure and function [31]—in terms of a system's invariant sets. Some use eigenvalues of linearization at an attracting equilibrium, which measure resilience to small perturbations in terms of characteristic return times [23]. Others, concerned with resilience to potentially large perturbations, measure the size of a domain of attraction [12], e.g. via n-dimensional domain volume [18] or distance from attractor to domain boundary [2]. Both eigenvalues and domain size lack information regarding transient dynamics within a domain of attraction, so neither fully reveals how a system will respond to continuous disturbances that drive the state away from invariant sets.

One can directly model a system's response to continuous disturbances by adding nonautonomous perturbations (controls) to the vector field, turning $x' = f(x)$ into $x' = f(x) + g(t)$. Proceeding in analogy to the theory developed in [16] for maps, we monitor the sets reachable under increasing bounds on the functions g. The critical bound at which perturbed trajectories first escape a domain of attraction defines a feature we call *intensity of attraction*. If domains of attraction delineate regions of distinct system structure/function[1] , then intensity quantifies an attractor's resilience to vector field perturbations according to the resilience definition from [31]. Unlike many existing

[1] As noted in [33], system function needn't always align with domains of attraction

measures of resilience that are based on a system's invariant sets, intensity reflects the strength of transient dynamics within a domain of attraction.

In chapter 5, we will relate an attractor's intensity to its continuation properties in the space of autonomous vector fields. This connection requires a *strict* bound on control functions. We therefore begin by introducing notation and preliminary facts for this $\|g\|_\infty \lneq r$ case.

4.1 Strictly Bounded Control Functions

To shift our attention to strictly bounded control functions, we change the closed ball $\overline{B_r(0)}$ in definition 3.5 to an open ball $B_r(0)$.

Definition 4.1. Given a vector field f and a control bound $r > 0$, let

$$\begin{aligned} \Psi_{r;f} = \{(a,b,T) \mid \text{ for some } g \in B_r(0) \subset L^\infty([0,T],\mathbb{R}^n), \text{ a solution } x:[0,T] \to \mathbb{R}^n \\ \text{to } x' = f(x) + g(t) \text{ satisfies } x(0) = a, x(T) = b\} \end{aligned}$$

and let $\Psi^t_{r;f} = \{(a,b) \mid (a,b,t) \in \Psi_{r;f}\}$.

The time-t relation $\Psi^t_{r;f}$ can be interpreted as a set-valued map, as follows:

Definition 4.2. For $r > 0$, $a \in \mathbb{R}^n$, and $S \subset \mathbb{R}^n$, let

$$\begin{aligned} &\Psi^t_{r;f} : \mathbb{R}^n \to \mathscr{P}(\mathbb{R}^n) \text{ be given by} \\ &\qquad a \mapsto \Psi^t_{r;f}(a) = \{b \in \mathbb{R}^n \mid (a,b,t) \in \Psi_{r;f}\}, \\ \text{and} \quad &\Psi^t_{r;f} : \mathscr{P}(\mathbb{R}^n) \to \mathscr{P}(\mathbb{R}^n) \text{ be given by} \\ &\qquad S \mapsto \Psi^t_{r;f}(S) = \bigcup_{a \in S} \Psi_{r;f}(a). \end{aligned}$$

For notational convenience, when $r = 0$ let $\Psi^t_{0;f}(a) \equiv \varphi^t(a)$ and $\Psi^t_{0;f}(S) \equiv \varphi^t(S)$.

The subscript f on $\Psi_{r;f}$ and $\Psi^t_{r;f}$ will be omitted when the vector field is clear or general. One can confirm that Ψ_r satisfies the identity and semigroup properties of a multiflow using arguments similar to those in the proof of Proposition 3.6. However, as the following example shows, Ψ_r need not be closed.

Example 4.3. With $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = -x$ and $g : \mathbb{R} \to \mathbb{R}$, the initial value problem

$$x' = -x + g(t), \qquad x(0) = 0 \tag{4.1}$$

has analytic solution $x(t) = e^{-t} \int_0^t e^s g(s) ds$. The sequence of constant controls $g_n = \frac{nr}{n+1}$ corresponds to a sequence of triplets $\{(0, g_n(1 - e^{-t}), t)\}_{n=1}^{\infty} \subset \Psi_{r;f} \subset \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$. This sequence has $(0, r(1 - e^{-t}), t)$ as a limit point. Yet the limit point cannot belong to $\Psi_{r;f}$, as $\|g\|_\infty < r$ implies $x(t) = e^{-t} \int_0^t e^s g(s) ds < e^{-t} \int_0^t e^s r ds = r(1 - e^{-t})$ for any $t \geq 0$. Because it does not contain all its limit points, $\Psi_{r;f}$ is not closed. //

4.2 Reachable Sets

We use $P_{r;f}(S)$ to denote the collection of states that can be reached in forward time from a set $S \subset U$ by modifying a vector field f with strictly r-bounded control (if $r > 0$) or no control (if $r = 0$):

Definition 4.4. $P_{r;f}(S) = \bigcup_{t \geq 0} \Psi_{r;f}^t(S)$, or equivalently, $P_{r;f}(S) = \bigcup_{t \geq 0} \bigcup_{a \in S} \bigcup_{g \in B_r(0)} \varphi_g^t(a)$.

When no confusion could result by omitting the vector field f, we write $P_r(S) = P_{r;f}(S)$.

Example 4.5. If I is invariant under the flow generated by f, then $P_{0;f}(I) = I$. //

Of principal interest are sets reachable from an attractor A. These can be found easily for one-dimensional systems, as the following proposition shows.

Proposition 4.6. *Consider the one-dimensional system $x' = f(x)$ with $f(0) = 0$, $f > 0$ on $(-\epsilon, 0)$, and $f < 0$ on $(0, \epsilon)$ for some $\epsilon > 0$. The reachable set from the attractor $A = \{0\}$ is $P_r(A) = (a, b)$ where*

$$a = \begin{cases} \sup\{x < 0 \mid f(x) = r\} & \textit{if } f(x) = r \textit{ for some } x < 0 \\ -\infty & \textit{otherwise} \end{cases}$$

$$\textit{and} \quad b = \begin{cases} \inf\{x > 0 \mid f(x) = -r\} & \textit{if } f(x) = -r \textit{ for some } x > 0 \\ \infty & \textit{otherwise} \end{cases}.$$

Proof: We begin with the inclusion $(a, b) \subset P_r(A)$. Let c be any point in $[0, b)$ and let $-m$ be the minimum value of f on $[0, c]$. This minimum must be strictly greater than $-r$ by construction of b. Let g be the constant function $g(t) = (m+r)/2$, so that $0 < m < g(t) < r$. Then $\|g(t)\|_\infty < r$. Also, since $f(x) \geq -m$ on $[0, c]$, $g(t) > m$ implies $f(x) + g(t) > 0$ on $[0, c]$. Using separation of variables, one may confirm that $c = \varphi_g^t(0)$ for finite, non-negative time $t = \int_0^c \frac{dx}{f(x)+(m+r)/2}$. A similar argument gives that any point in $(a, 0]$ is reachable in finite time by a constant control strictly bounded by r. Definitions 4.1, 4.2 and 4.4 then imply that $(a, b) \subset P_r(A)$.

The reverse inclusion follows trivially if $a = -\infty$ and $b = \infty$. Suppose that $b < \infty$. Since f is continuous, $f(b) = -r$ and $f(b) + r = 0$. Therefore, choosing $g(t) \equiv r$ in equation 3.1 yields an autonomous flow with b as an equilibrium (Figure 4.1).

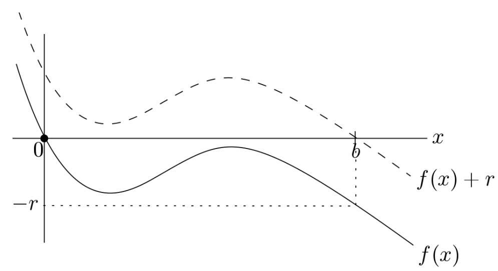

Figure 4.1: Augmenting a one-dimensional vector field by constant r.

The Lipschitz property of f implies that $f(x) + r \leq -L(x - b)$ on $[0, b]$. We use this fact and a comparison of solutions to the initial value problems

$$x' = -L(x - b), \;\; x(0) = 0 \tag{4.2}$$

and

$$y' = f(y) + g(t), \;\; y(0) = 0 \tag{4.3}$$

to show that $\varphi_g^t(0) < b$ when $\|g\|_\infty < r$.

For brevity, let $x(t)$ denote the solution to (4.2) and $y(t)$ denote the solution to (4.3) for a fixed g.

Assume for the sake of contradiction that there exists a $t > 0$ and $g \in L^\infty([0, t], \mathbb{R}^n)$ such that $y(t) \geq b$. Let T denote the minimum positive time t at which $y(t) = b$.

(That T is well-defined follows from continuity of $y(t)$ and $y(0) \neq b$.) Since $x(t)$ remains strictly bounded below b and both paths are continuous, the inequality $y(t) > x(t)$ must hold on some time interval $[t_*, T]$ with $0 \leq t_* < T$. In fact, one may choose t_* so that $x(t_*) = y(t_*)$. For $s \in [t_*, T]$, these inequalities must then hold:

$$f(y(s)) + g(s) < f(y(s)) + r \leq -L(y(s) - b) \leq -L(x(s) - b) \tag{4.4}$$

The second inequality follows from the Lipschitz property of $f(\cdot) + r$, which gives that $f(y(s)) + r \leq |(f(y(s)) + r) - (f(b) + r)| \leq L|y(s) - b| = -L(y(s) - b)$ for $s \in [t_*, T]$.

On the other hand, the assumptions $x(T) < y(T)$ and $x(t_*) = y(t_*)$ imply that

$$\int_{t_*}^{T} -L(x(s) - b)ds < \int_{t_*}^{T} [f(y(s)) + g(s)]\, ds \tag{4.5}$$

Comparing inequalities (4.4) and (4.5) yields the desired contradiction. Hence the solution $y(t)$ to IVP (4.3) must remain strictly bounded below b in forward time; i.e. $\varphi_g^t(0) \in (-\infty, b)$ for $\|g\|_\infty < r$ and $t \geq 0$. Definitions 4.1, 4.2 and 4.4 imply that $P_r(A) \subset (-\infty, b)$. In the case that $a > -\infty$, the inclusion $P_r(A) \subset (a, \infty)$ follows similarly. Together, the inclusions $P_r(A) \subset (-\infty, b)$ and $P_r(A) \subset (a, \infty)$ imply that $P_r(A) \subset (a, b)$ when a, b, or both are finite. □

In the case of a one-dimensional linear system, Proposition 4.6 implies that the set reachable from the origin scales in direct proportion to the control bound and in inverse proportion to the eigenvalue magnitude:

Corollary 4.6.1. *For the one-dimensional, linear system $x' = -\lambda x$ ($\lambda > 0$) with global attractor $A = \{0\}$, the reachable set from A for $r > 0$ is $P_r(A) = (-\frac{r}{\lambda}, \frac{r}{\lambda})$.*

Reachable sets are, in general, harder to compute than in Proposition 4.6 and Corollary 4.6.1. The following example shows that the computation can be non-trivial even for a diagonal, two-dimensional linear system.

Example 4.7. The system

$$x' = -x \tag{4.6}$$

$$y' = -2y \tag{4.7}$$

on $\mathbb{R}^2$ has global attractor $A = \{(0,0)\}$. Considering each variable separately, corollary 4.6.1 indicates that $P_1(A)$ is contained in the rectangular region $R = (-1,1) \times (-\frac{1}{2}, \frac{1}{2})$. Furthermore, the family of constant control functions $\{g_{\theta,c}(t) = c\cos\theta + c\sin\theta \mid \theta \in [0, 2\pi), 0 \leq c < 1\}$ sends the origin asymptotically to points on ellipses $x^2 + 4y^2 = c^2$. Therefore, the reachable set $P_1(A)$ contains the open region U enclosed by the ellipse $E = \{(x,y) \mid x^2 + 4y^2 = 1\}$. Control functions with magnitude 1 directed opposite to the vector field also drive trajectories asympototically from U to E.

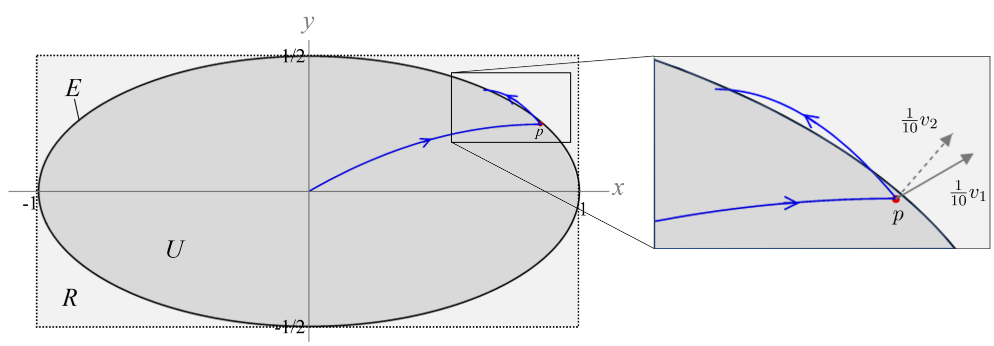

Figure 4.2: Inexact bounds on the reachable set for a 2D linear attractor

Because the vector field has magnitude 1 on E, one might expect that $P_1(A) = U$. But the vector field is not normal to E except at the vertices of the ellipse. Therefore, adjusting a control function's direction to push perpendicular to E can drive a solution outside U. Figure 4.2 illustrates this phenomenon for piecewise constant control

$$g(t) = \begin{cases} v_1 & \text{if } 0 \leq t < 10 \\ v_2 & \text{if } 10 \leq t \end{cases} \tag{4.8}$$

where $v_1 = 0.99 \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$ and $v_2 = 0.99 \begin{bmatrix} \sqrt{3/7} \\ 2/\sqrt{7} \end{bmatrix}$ have magnitude 0.99 but different directions. We therefore have the bounds $U \subsetneq P_1(A) \subset R$. Athough these bounds follow directly from analytic computation of trajectories and corollary 4.6.1, a closed form for $P_1(A)$ does not. //

Numerical algorithms for computing reachable sets are an active topic of research in the fields of control systems, differential inclusions, and optimization. The introductory piece [14] outlines fundamental discretization challenges nicely. Existing approaches include include set-valued Euler schemes [3, 25, 26, 28], optimal control algorithms [24, 1], and level-set methods ([9] and references therein).

Though the approximation of reachable sets can be involved, analytically they obey nice nesting properties. The following lemma, which follows immediately from Definitions 4.1, 4.2, and 4.4, states that the real number ordering of control bounds induces an order by inclusion on reachable sets:

Lemma 4.8. *If $r < r'$ then $P_r(S) \subset P_{r'}(S)$.*

A stronger statement, whose proof appears in Chapter 5, holds when the set S is an attractor A:

Proposition 4.9. *$P_r(A) \searrow A$ as $r \searrow 0$, in the sense that*

(1) $r < r'$ implies $P_r(A) \subset P_{r'}(A)$

(2) $A = \bigcap_{r>0} P_r(A)$, and

(3) for any neighborhood V of A, there is an $r > 0$ such that $P_r(A) \subset V$.

We delay the proof of Proposition 4.9 until Section 5.4, after the introduction of attractor blocks in Section 5.1.

4.3 Definition of Intensity

Intensity of attraction, introduced by McGehee for maps [16], carries over naturally to the flow setting by replacing sets reachable from an attractor under ϵ pseudo-orbits with sets reachable from an attractor under strictly r-bounded control.

Definition 4.10. The *intensity of attraction* of an attractor A is

$$\mu(A) = \sup\{r \geq 0 : P_r(A) \subset K \subset \mathcal{D}(A) \text{ for some compact } K \subset \mathbb{R}^n\} \tag{4.9}$$

In other words, intensity of attraction reflects the control magnitude $\|g\|_\infty$ necessary to escape from all compact subsets of an attractor's domain.

Proposition 4.11. *For A an attractor, $\mu(A) > 0$.*

Our proof of Proposition 4.11, like that of Proposition 4.9, relies on the existence of attractor blocks (Lemma 5.7) and appears in Section 5.4.

The value $\mu(A)$ depends on the norm $\|\cdot\|$ employed on $\mathbb{R}^n$, as example 4.12 illustrates.

Example 4.12. Consider the system

$$x' = \frac{\sqrt{2}}{4}(x+y)^2 - x \tag{4.10}$$

$$y' = \frac{\sqrt{2}}{4}(x+y)^2 - y \tag{4.11}$$

whose phase portrait is given in Figure 4.3a. Rotating coordinates by $\pi/4$ to $u = \frac{\sqrt{2}}{2}(x+y)$ and $v = \frac{\sqrt{2}}{2}(y-x)$ gives

$$u' = u(u-1) \tag{4.12}$$

$$v' = -v. \tag{4.13}$$

In these coordinates it is clear that $A = \{(0,0)\}$ is an attractor with domain of attraction $\mathcal{D}(A) = \{(u,v) \mid u < 1\}$. The uncoupled form of equations (4.12) and (4.13) and the geometry of $\mathcal{D}(A)$ allow us to restrict attention to the invariant line $v = 0$ when calculating $\mu(A)$. On the interval $0 \leq u \leq 1$, the maximum recovery rate towards the origin is $1/4$, achieved by the vector m at $u = 1/2$. A control function must push with magnitude greater than $1/4$ in the direction of $-m$ to steer a trajectory from A to $\mathcal{D}(A)^C$. The critical control bound needed for this push, $\mu(A)$, depends on the norm on (x,y) space. Under the Euclidean norm (the 2-norm), the intensity is $\mu_2(A) = 1/4$ (Figure 4.3c). For a given bound on controls, the 1-norm admits a smaller push in the

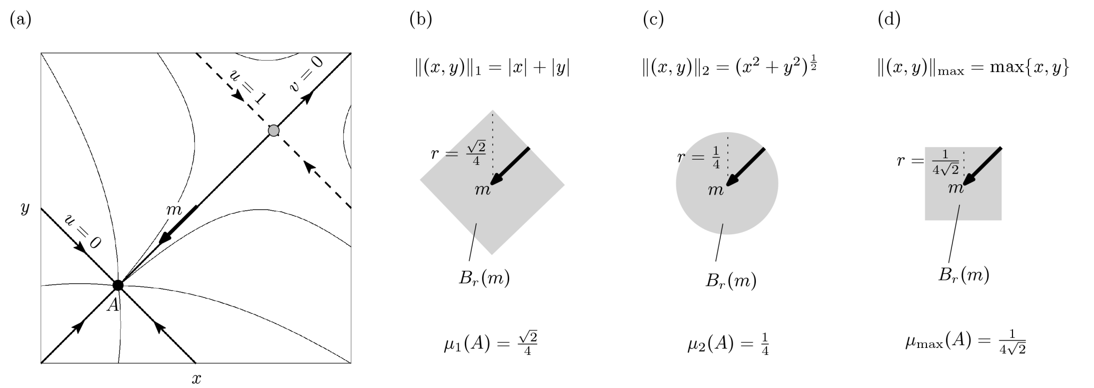

Figure 4.3: Dependence of intensity on norm / metric

direction $-m$ and the max norm admits a larger push, relative to the 2-norm. The intensity of A under the 1-norm is $\mu_1(A) = \sqrt{2}/4$ (Figure 4.3b) and the intensity under the max norm is $\mu_{\max}(A) = 1/4\sqrt{2}$ (Figure 4.3d). More generally, a straightforward calculation gives that under the p-norm $\|(x,y)\|_p = (x^p + y^p)^{1/p}$, the intensity of A is $\mu_p(A) = 2^{\left(\frac{1}{p}-\frac{1}{2}\right)} \cdot \frac{1}{4}$. //

In specific applications, Carpenter and colleagues' mantra "resilience to what" [4] may help guide norm choice. We anticipate the max norm will be appropriate when bounds on vector field perturbations are known componentwise rather than jointly. As in Example 4.12, one expects $\mu_p(A) \leq \mu_q(A)$ when $p > q$. Therefore, choosing a max norm may give the most conservative estimate of resilience.

We turn to a comparison between intensity and a closely related concept examined by Colonius and Kliemann [6]: the lowest point of discontinuity in a map from control bound to the reachable set (control set) of an attractor. Example 4.13 demonstrates that this first discontinuity may differ from the intensity of the attractor.

Example 4.13. The one-dimensional system with vector field $f(x) = \frac{3}{4}x^4 - x^3 - 3x^2 - 1$ has an attractor $A = \{x_0\}$ and repeller $R = \{x_1\}$ corresponding to the roots x_0 and x_1 of the quartic (Figure 4.4a). The domain of attraction of A is $\mathcal{D}(A) = (-\infty, x_1)$. Consider the map $C : [0,\infty) \to \mathscr{P}(\mathbb{R})$ given by $r \mapsto P_r(A)$. Figure 4.4b depicts the graph of C. As the control bound r increases from zero, the first discontinuity in C occurs at $r = 2.25$, corresponding to the local minimum of f at $(-1, -2.25)$. However, reachable sets remain bounded within $\mathcal{D}(A)$ for small enough $r > 2.25$. It is not until the second discontinuity at $r = 9$, corresponding to the global minimum of f at $(2, -9)$, that reachable sets escape $\mathcal{D}(A)$. Hence the intensity $\mu(A) = 9$ exceeds the value of r at the first discontinuity ($\rho^* = 2.25$ in the notation of [6]). //

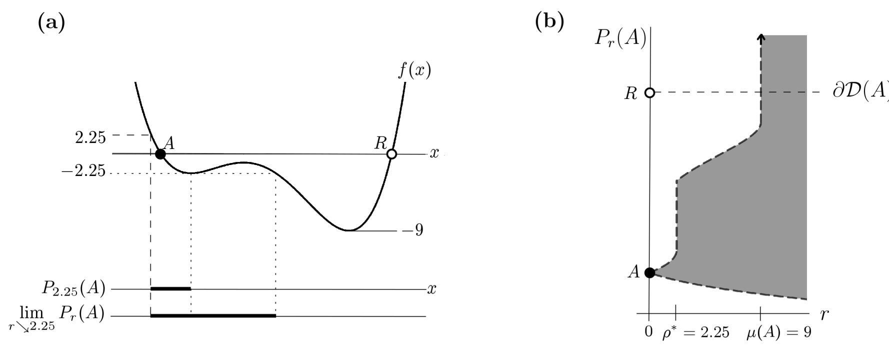

Figure 4.4: A comparison between intensity and first discontinuity of reachable sets

Example 4.13 shows that it is possible for reachable sets to expand discontinuously without escaping a domain of attraction. Whether the reverse is possible—reachable sets escaping a domain of attraction without expanding discontinuously—is an interesting question not pursued further here.

We close this section with a technical lemma used in chapter 5.

Lemma 4.14. *The set* $R = \{r \geq 0 : P_r(A) \subset K \subset \mathcal{D}(A)\}$ *contains the interval* $[0, \mu(A))$.

Proof: Suppose that $0 \leq r < \mu(A)$. Then there must exist an $r' > r$ such that $P_{r'}(A) \subset K \subset \mathcal{D}(A)$, else $r \geq \mu(A)$. Lemma 4.8 implies $P_r(A) \subset P_{r'}(A)$. So $P_r(A) \subset K \subset \mathcal{D}(A)$, and $r \in R$. □

4.4 Applications of Intensity

Example 4.15 illustrates how intensity of attraction detects transient recovery dynamics that other resilience metrics miss.

Example 4.15. A previous work by the present author and colleagues [20] highlighted aspects of resilience neglected by common resilience quantification methods by comparing the following population models.

$$\frac{dx}{dt} = x\left(1 - \frac{x}{100}\right)\left(\frac{x}{20} - 1\right) \tag{4.14}$$

$$\frac{dx}{dt} = x\left(1 - \frac{x}{100}\right)\left(\frac{x}{20} - 1\right)(0.0002x^2 - 0.024x + 1.4) \tag{4.15}$$

Figure 4.5, adapted from Figure 1 of [20], plots the vector fields corresponding to (4.14) (solid) and (4.15) (dashed). Both models have a stable equilibrium at $x = 0$, an unstable equilibrium (Allee threshold) at $x = 20$, and a stable equilibrium (carrying capacity) at $x = 100$. Further, the second factor in equation (4.15) has been chosen to preserve derivatives at $x = 20$ and $x = 100$. Therefore, two common methods for quantifying resilience—distance from a stable equilibrium to the boundary of its domain of attraction and eigenvalues of linearization [19]—indicate identical resiliences at carrying capacity between the two populations. Each population can withstand a one-time reduction in population size of up to 80 units, and each exhibits similar return rates when perturbed slightly away from 100.

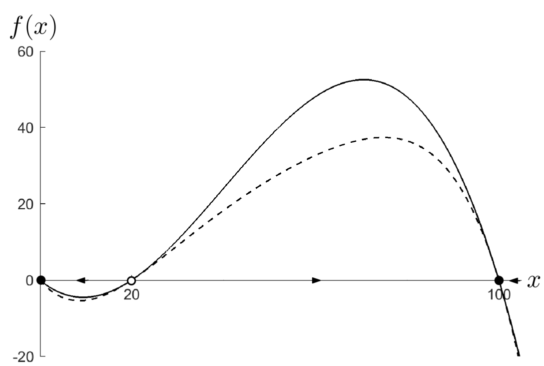

Figure 4.5: Growth rates of two Allee populations [adapted from Figure 1 of [20]].

However, the curves in Figure 4.5 indicate that a population with solid curve dynamics can reach higher maximum growth rates than a population with dashed curve dynamics. Intensity of attraction detects this difference in transient recovery strength: using

Proposition 4.6 and Definition 4.10, one can confirm that $\mu(\{100\})$ matches the maximum value of f on $[20, 100]$ for each model, so $\{100\}$ has greater intensity of attraction under (4.14) than under (4.15). This difference in intensities means that the population (4.14) can persist under stronger continual perturbations (for example, ongoing variability in birth and mortality due to climate or disease) that drive it below carrying capacity. //

As suggested by example 4.15 and Proposition 4.6, intensity of attraction can be determined in a one-dimensional system from the local minima and maxima of f. The utility of Definition 4.10 lies not in one-dimensional applications but in the fact that intensity continues to be well-defined in systems of higher dimension where maximum recovery rates are not immediately apparent. Example 4.16 illustrates such an application.

Example 4.16. We compute intensity of attraction for a spiral sink and a stable limit cycle in the predator-prey model [15, 27]

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{K}\right) - ky(1 - e^{-cx}) \tag{4.16}$$

$$\frac{dy}{dt} = -by + \beta y(1 - e^{-fx}). \tag{4.17}$$

In this model, the density x of a prey species grows logistically in the absence of predator, with intrinsic growth rate a and carrying capacity K. This growth is offset by predation, parametrized by k and c. In the absence of prey, the predator density y declines at a rate parametrized by b. The benefits of prey consumption, parametrized by β and f, can offset this mortality to yield net growth in predator density. The parameter choices $a = 1$, $K = 3$, $k = 0.5$, $c = 1.5$, $b = 0.5$, $\beta = 1$, and $f = 0.5$ (in arbitrary units) yield a spiral sink A_1 in the first quadrant for $K = 3$ (Figure 4.6a). As the prey carrying capacity K grows, a Hopf bifurcation occurs; setting $K = 4$ yields a stable limit cycle A_2 in the first quadrant (Figure 4.6c).

Rosenzweig used this predator-prey system to illustrate a "paradox of enrichment": adding nutrients to an ecosystem (thus increasing K) destabilizes the ecosystem [27]. His analysis was restricted to the dynamics near the rest point in the first quadrant,

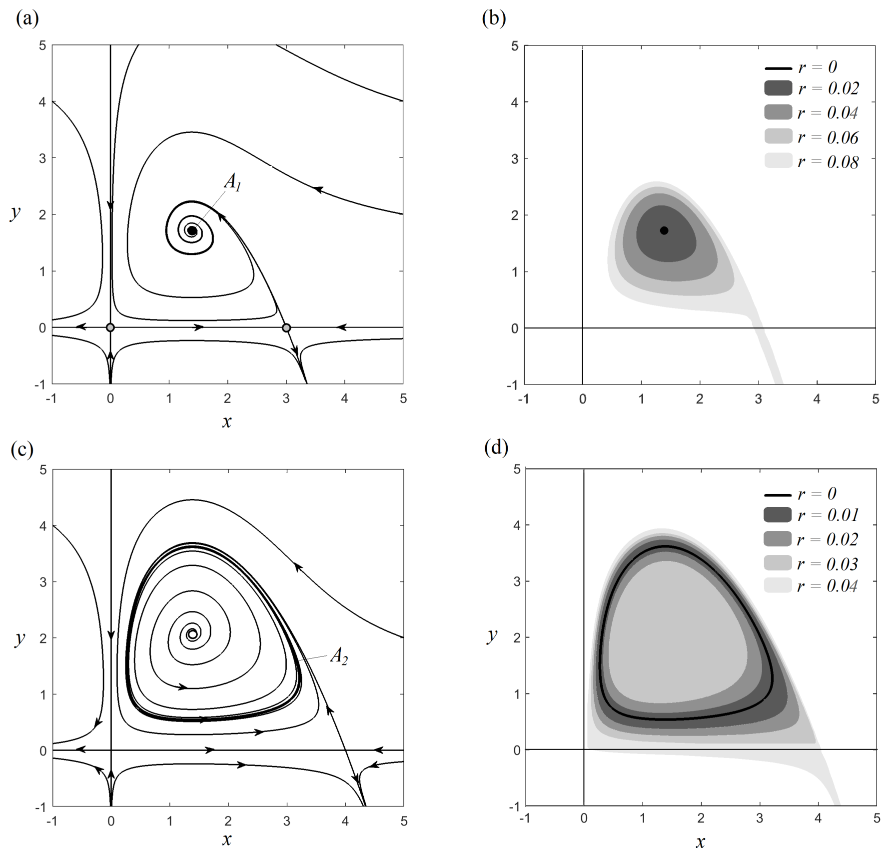

Figure 4.6: Reachable sets for a predator-prey equilibrium (a,b) and limit cycle (c,d)

which indeed turns from stable to unstable as K increases. But in what sense is the periodic orbit A_2 in Figure 4.6c "destabilizied" relative to the spiral sink A_1 in Figure 4.6a? Both invariant sets are stable by standard mathematical definitions. One answer can be seen by eye: as predator and prey densities oscillate along the periodic orbit, their minimum densities are much smaller than at the stable rest point. This suggests a greater risk of extinction via random fluctuations.

A second answer comes through computing the intensities of the spiral sink and the periodic orbit. As Figure 4.6b indicates, the sets reachable from the spiral sink remain bounded within its domain of attraction (the first quadrant) for $r \leq 0.06$. For $r = 0.08$, the reachable set spills over the x-axis. From this picture we estimate

$0.6 < \mu(A_1) < 0.8$. Figure 4.6d illustrates sets reachable from the periodic orbit A_2. These remain bounded within its domain of attraction (the first quadrant minus the unstable rest point) for $r \leq 0.02$; for $r = 0.03$, the reachable set includes the unstable rest point. Hence $0.02 < \mu(A_2) < 0.03$. If we instead consider the attractor A_3, consisting of the periodic orbit A_2 and the region it encloses, we get a slightly higher intensity $0.03 < \mu(A_3) < 0.04$. Regardless of whether we consider A_2 or A_3, the intensity estimates agree with Rosenzweig's argument: the attractor in the first quadrant has a lower intensity of attraction when $K = 4$ than when $K = 3$. //

Chapter 5

Continuation of Attractors

Errors are inherent in ODE models of physical and biological systems, from the simplifying assumptions that shape functional forms to the data that inform parameter choices. Nonetheless, if the resulting model's qualitative features can be shown to persist in nearby systems, it seems plausible that they represent some essential aspect of reality. Classical results regarding structurally stable systems and isolated invariant sets give persistence of qualitative model behavior under sufficiently small perturbations. In this chapter we push beyond "sufficiently small" to explore just how different a vector field can become while retaining a continuation of an original attractor. Section 5.1 introduces basic definitions and results regarding isolated invariant sets in general and attractors in particular. Section 5.2 establishes a connection between reachable sets and attractor blocks, which we exploit in section 5.3 and to show that intensity of attraction gives a lower bound on attractor continuation distance. Theorem 5.14, the main result of this chapter, grounds the abstract question of attractor persistence in infinite dimensional vector field space to the study of a single system's reachable sets, which can be approximated numerically. We conclude in section 5.4 with a proof of the semicontinuity of attractors based on reachable sets.

5.1 Attractors as Isolated Invariant Sets

Attractors are a special type of isolated invariant set, a useful object in the study of robust features of flows. The theory of isolated invariant sets has been developed in the settings of flows on smooth manifolds [8, 32] and locally compact metric spaces [21]. Definitions 5.1 and 5.2 tailor those of [21] to flows on $\mathbb{R}^n$.

Definition 5.1. An *isolating neighborhood* for a flow φ is a compact set $N \subset \mathbb{R}^n$ whose invariant part $\mathrm{Inv}(N, \varphi) \equiv \{x \in N \mid \varphi(\mathbb{R}, x) \subset N \text{ for all } t \in \mathbb{R}\}$ lies in the interior of N.

Definition 5.2. A set $I \subset \mathbb{R}^n$ is an *isolated invariant set* if $I = \mathrm{Inv}(N)$ for some isolating neighborhood N.

Remark 5.3. Definitions 2.7 and 5.2 imply that an attractor A is an isolated invariant set; the isolating neighborhood N can be constructed via Lemma 2.1 as a compact set nested between A and $\hat{N}$, where $\hat{N}$ is a neighborhood of A such that $A = \omega(\hat{N})$.

An important consequence of Definition 5.2 is that isolating neighborhoods continue as such under sufficiently small perturbations to the flow, though the maximal invariant set in the interior may change [21]. Homotopic and homological Conley indices built from an isolating neighborhood can yield coarse information about the isolated invariant set inside. These allow one to deduce topological features of an isolated invariant set that persist as it continues in nearby systems. In the words of Conley and Easton [8],

> "The main theorem of this report is [that] ... any invariant set that is the maximal one in some neighborhood of itself is the maximal one in some isolating block [a special isolating neighborhood, see below]. This result together with the perturbation result and the fact that the form of the block determines properties of the invariant set indicate that a useful step in the qualitative analysis of flows might be to search for isolated invariant sets."

When the flow and boundary of an isolating neighborhood interact nicely, the neighborhood earns the aforementioned name *isolating block*. Various definitions of "nicely" have been used (see, for example [8, 32, 7, 21]). Given our interests we focus on attractor blocks—isolating blocks associated with attractors.

Definition 5.4. An *attractor block* B for a flow $\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonempty, compact subset of $\mathbb{R}^n$ satisfying $\varphi^t(B) \subset \mathrm{int}(B)$ for all $t > 0$.

Lemma 5.5. *For B an attractor block, $\omega(B)$ is an attractor.*

Proof: $\omega(B)$ must lie interior to B because $\varphi^t(B) \subset \text{int}(B)$ for all $t > 0$. Therefore $\omega(B)$ is the omega limit set of its neighborhood B. Further, $\omega(B)$ is a closed subset of compact B and hence compact, and is easily seen to be invariant. By Definition 2.7, it is an attractor. □

Definition 5.6. If B is an attractor block and $\omega(B) = A$, we call B an attractor block *associated with* A.

We have already seen in Lemma 5.5 that any attractor block is associated with an attractor in its interior. Lemma 5.7 states that the converse also holds.

Lemma 5.7. *For any attractor A, there exists an attractor block B associated with A.*

Proofs of the corresponding general statement regarding isolated invariant sets and isolating blocks appear in [8, 32]. We sketch here a construction of an attractor block using a Lyapunov function. In the notation of Remark 5.3, let $A = \omega(N)$ and let $\hat{N}$ be an isolating neighborhood with $\text{Inv}(\hat{N}) = A$ and $\hat{N} \subset N$. Wilson and Yorke have shown there is an open neighborhood Ω of A in $\hat{N}$ and a monotone Lyapunov function $L : \Omega \to \mathbb{R}$ such that $L(x) = 0$ if $x \in A$ and $\frac{d}{dt}L(x(t)) < 0$ if $x \in \Omega - A$ ([32] Theorem 2.2). It follows that for sufficiently small ϵ, the preimage $B = L^{-1}([0, \epsilon])$ is a compact neighborhood of A inside Ω. Strict monotonicity of the Lyapunov function implies $\varphi^t(B) \subset \text{int}(B)$ for all $t > 0$. Further, $A \subset B \subset \hat{N} \subset N$ implies $\omega(B) = A$. Hence B is an attractor block associated with A.

5.2 Attractor Blocks from Reachable Sets

In this section we show that certain reachable sets can be used to construct attractor blocks; furthermore, these blocks persist as such not just for sufficiently small perturbations to the vector field, but for perturbations smaller than r, the metric bound on control. Before stating these results we establish a definition and technical lemma.

Fixing all but the third argument in the controlled flow function (definition 3.2) yields a map from control functions to trajectory endpoints.

Definition 5.8. Let $M_{x_0,\tau;f} : L^\infty([0,\tau],\mathbb{R}^n) \to \mathbb{R}^n$ be given by $g \mapsto \varphi(\tau, x_0, g; f)$. The subscripts on M may be omitted when f, x_0 and τ are clear from context, or general.

Lemma 5.9. *M is an open map.*

Proof: It suffices to show that M maps any basis element in the metric topology on $L^\infty([0,\tau],\mathbb{R}^n)$ to an open set in $\mathbb{R}^n$, since this implies that if V is an open set in $L^\infty([0,\tau],\mathbb{R}^n)$, $V = \bigcup B_i$ for some basis elements B_i, so $M(V) = M(\bigcup B_i) = \bigcup M(B_i)$, a union of open sets.

A basis element of the metric topology on $L^\infty([0,\tau],\mathbb{R}^n)$ is an open ball; call it $B_r(g) \equiv \{h \in L^\infty([0,\tau],\mathbb{R}^n) : \|h - g\|_\infty < r\}$. To show that $M(B_r(g))$ is open, we will show that for any $h \in B_r(g)$ there exists an ϵ_h sufficiently small so that the open ϵ_h-ball around $M(h)$ is also in the image $M(B_r(g))$. This will imply that $M(B_r(g)) \subset \bigcup_{h\in B_r(g)} B_{\epsilon_h}(M(h)) \subset M(B_r(g))$, implying $M(B_r(g))$ is equal to a union of open sets, and hence open.

Fix $h \in B_r(g)$ and consider a point $M(h)+v$ in $\mathbb{R}^n$. We derive an ϵ_h such that $\|v\| < \epsilon_h$ implies the existence of a $k \in B_r(g)$ such that $M(k) = M(h) + v$. First we construct a path from x_0 to $M(h) + v$. Let $x_h(t) = \varphi(t, x_0, h)$ and let

$$\tilde{x}(t) = x_h(t) + t\frac{v}{\tau}.$$

Then $\tilde{x}(0) = x_h(0) = x_0$ and $\tilde{x}(\tau) = x_h(\tau) + v = M(h) + v$, so $\tilde{x}$ is a path from x_0 to $M(h) + v$. The velocity vectors along the path $\tilde{x}$ (defined almost everywhere) are

$$\frac{d}{dt}\tilde{x}(t) = \frac{d}{dt}\left[x_h(t)\right] + \frac{v}{\tau} = f(x_h(t)) + h(t) + \frac{v}{\tau}.$$

The control $k(t)$ required to achieve the path $\tilde{x}(t)$ as a solution to $x' = f(x)+k(t)$ is the difference between the velocity vectors along the path and the underlying vector field, so

$$k(t) = \frac{d}{dt}\tilde{x}(t) - f(\tilde{x}(t)) = f(x_h(t)) + h(t) + \frac{v}{\tau} - f(\tilde{x}(t)).$$

We aim to show that for $|v|$ sufficiently small, $k \in B_r(g)$; that is, $\|k - g\|_\infty < r$. For almost every $t \in [0, \tau]$,

$$\begin{aligned}\|k(t) - g(t)\| &= \|f(x_h(t)) + h(t) + \frac{v}{\tau} - f(\tilde{x}(t)) - g(t)\| \\ &\le \|f(x_h(t)) - f(\tilde{x}(t))\| + \|h(t) - g(t)\| + \frac{\|v\|}{\tau} \\ &\le L\|x_h(t) - \tilde{x}(t)\| + \|h(t) - g(t)\|_\infty + \frac{\|v\|}{\tau} \\ &\le L\,\|v\| + \|h - g\|_\infty + \frac{\|v\|}{\tau} \\ &= \|v\|\,(L + \frac{1}{\tau}) + \|h - g\|_\infty\end{aligned}$$

Let $\epsilon_h = \frac{r - \|h-g\|_\infty}{L + 1/\tau}$. (Note that $\|h - g\|_\infty < r$ because $h \in B_r(g)$.) Then $\|v\| < \epsilon_h$ implies that $\|k(t) - g(t)\|_\infty < r$. Because there exists an ϵ_h-ball about any point in $M(B_r(g))$ that is also in the image $M(B_r(g))$, the image of any basis element under M is open, and the proof is complete. □

Lemmas 5.10 and 5.11 present the main results in this section.

Lemma 5.10. *If A is an attractor and if $P_r(A)$ is contained in a compact subset of $\mathcal{D}(A)$, then $\overline{P_r(A)}$ is an attractor block associated with A.*

We omit proof of Lemma 5.10 because it follows directly from Lemma 5.11. The latter gives conditions under which $\overline{P_r(A)}$ persists as an attractor block through perturbations of the original vector field.

Lemma 5.11. *If A is an attractor for $x' = f(x)$ and if $P_r(A)$ is contained in a compact set in $\mathcal{D}(A)$, then for any second globally Lipchitz and bounded vector field $\hat{f} : U \to \mathbb{R}^n$ satisfying $\left\|f - \hat{f}\right\|_{sup} < r$, the set $\overline{P_r(A)}$ (defined using f) is also an attractor block for the flow $\hat{\varphi}$ generated by $\hat{f}$.*

Proof: As a closed subset of the compact set $\mathcal{D}(A)$, $\overline{P_r(A)}$ is compact. To confirm that $\overline{P_r(A)}$ is an attractor block for $\hat{\varphi}$, it suffices to show that for any $\tau > 0$,

$$\hat{\varphi}^\tau(\overline{P_r(A)}) \overbrace{\subset}^{\text{(I)}} \Psi^\tau_{r;f}(\overline{P_r(A)}) \overbrace{\subset}^{\text{(II)}} \operatorname{int}(\overline{P_r(A)}). \tag{5.1}$$

For inclusion (I), we first establish that $\hat{\varphi}^\tau(x_0) \in \Psi^\tau_{r;f}(x_0)$ for any point $x_0 \in \mathbb{R}^n$. The function $\hat{x} : [0, \tau] \to \mathbb{R}^n$ given by $\hat{x}(t) = \hat{\varphi}^t(x_0)$ solves the initial value problem $[x' = \hat{f}(x),\ x(0) = x_0]$. Therefore $\hat{x}$ is the solution to the initial value problem $[x' = f(x) + g(t),\ x(0) = x_0]$, where $g : [0, \tau] \to \mathbb{R}^n$ is given by $g(t) = \hat{f}(\hat{x}(t)) - f(\hat{x}(t))$. By hypothesis, $\|g\|_\infty \leq \left\| f(x) - \hat{f}(x) \right\|_{\sup} < r$. Definitions 4.1 and 4.2 give that $\hat{\varphi}^\tau(x_0) = \hat{x}(\tau) \in \Psi^\tau_{r;f}(x_0)$. Inclusion (I) for the set $\overline{P_r(A)}$ follows directly from Definitions 2.2 and 4.2.

We argue inclusion (II) by contradiction. Suppose that $\Psi^\tau_{r;f}(\overline{P_r(A)}) \not\subset \operatorname{int}(\overline{P_r(A)})$. Then there exists a point $b \in \Psi^\tau_{r;f}(\overline{P_r(A)})$ such that $b \notin \operatorname{int}(\overline{P_r(A)})$. By Definitions 4.1 and 4.2, $b \in \Psi^\tau_{r;f}(\overline{P_r(A)})$ implies existence of a control $g \in B_r(0) \subset L^\infty([0, \tau], \mathbb{R}^n)$ and a point $a \in \overline{P_r(A)}$ such that $\varphi(t, a, g; f) = b$. Because $\|g\|_\infty$ is strictly bounded below r, there exists an $s > 0$ such that the open s-ball about g, $B_s(g) \subset L^\infty([0, \tau], \mathbb{R}^n)$, also lies within $B_r(0) \subset L^\infty([0, \tau], \mathbb{R}^n)$. Lemma 5.9 gives that $M_{a,\tau;f}$ maps $B_s(g)$ to an open neighborhood, U, of b. The second assumption $b \notin \operatorname{int}(\overline{P_r(A)}$ implies that U intersects $\overline{P_r(A)}^{\mathrm{C}}$ nontrivially, so there exists a control $h \in B_s(g)$ such that $\hat{b} \equiv \varphi(\tau, a, h; f) \in \overline{P_r(A)}^{\mathrm{C}}$. Because $\overline{P_r(A)}^{\mathrm{C}}$ is open and $\varphi(\tau, x, h; f)$ is continuous with respect to initial condition x, the inverse image of $\overline{P_r(A)}^{\mathrm{C}}$ under $\varphi(\tau, \cdot, h; f)$ is an open neighborhood V of a. V must intersect $P_r(A)$ nontrivially because $a \in \overline{P_r(A)}$. Let $\hat{a} \in P_r(A) \cap V$. Then the point $c = \varphi(\tau, \hat{a}, h; f)$ is not contained in $P_r(A)$ for $\hat{a} \in P_r(A)$ and $\|h\|_\infty < r$.

On the other hand, $\hat{a} \in P_r(A)$ implies that $(x_0, \hat{a}) \in \Psi^t_{r;f}$ for some $x_0 \in A$ and $t \geq 0$. Combined with the fact that $(\hat{a}, c) \in \Psi^\tau_{r;f}$, the semigroup property of $\Psi_{r;f}$ gives that $(x_0, c) \in \Psi^{t+\tau}_{r;f}$. Definition 4.2 implies that $c \in P_r(A)$, and this contradiction completes the proof. $\square$

5.3 Intensity and Continuation Distance

Suppose that when a vector field f_1 is perturbed to f_2, the original attractor A_1 shifts to another attractor A_2, with qualitatively similar features. In what sense has A_1 persisted through the perturbation? The following definition of continuation, adapted from [16], formalizes one notion of persistence.

Definition 5.12. Given an attractor A_1 for vector field f_1 and an attractor A_2 for vector field f_2, we say A_1 *continues immediately* to A_2 if there exists a set B that is an attractor block associated with A_1 under the flow generated by f_1 and is an attractor block associated with A_2 under the flow generated by f_2.

Remark 5.13. Because A_1 and its immediate continuation A_2 share a common attractor block, they must also share any topological properties gleaned from the attractor block.

The following theorem gives the main result of this chapter: intensity of attraction bounds from below the distance in vector field space over which that attractor has an immediate continuation.

Theorem 5.14. *If A is an attractor for vector field f with intensity $\mu(A)$, then for any second vector field $\hat{f}$ satisfying $|f - \hat{f}|_{sup} < \mu(A)$, A continues immediately to an attractor $\hat{A}$ for $\hat{f}$.*

Proof: If $\left\|f - \hat{f}\right\|_{\text{sup}} < \mu(A)$, then there exists a real number $r > 0$ such that $\left\|f - \hat{f}\right\|_{\text{sup}} < r < \mu(A)$. By Lemma 4.14, $P_r(A) \subset K \subset \mathcal{D}(A)$ for some compact set K. This implies, by Lemma 5.10, that $\overline{P_r(A)}$ is an attractor block associated with A for vector field f. Lemma 5.11 gives that $\overline{P_r(A)}$ is also an attractor block for vector field $\hat{f}$. Defining $\hat{A}$ to be the omega limit set of $\overline{P_r(A)}$ under $\hat{f}$, we have that $\overline{P_r(A)}$ is an attractor block associated with attractors A and $\hat{A}$ under their respective flows. Hence A continues immediately to $\hat{A}$, as claimed. $\square$

The following example illustrates continuation of the predator-prey limit cycle from Example 4.16.

Example 5.15. Consider the predator-prey system in Example 4.16, featuring a stable limit cycle A in the first quadrant. Based on reachable set computations with the Euclidean norm, we estimated $0.02 < \mu(A) < 0.03$. By Theorem 5.14, A should continue immediately to an attractor $\hat{A}$ for any second vector field within 0.02 of the original (equations 4.16 and 4.17) in Euclidean sup-norm. Figure 5.1 illustrates the immediate continuation $\hat{A}$ for three forms of vector field perturbation: (a) adding 0.02 to dx/dt, (b) subtracting 0.02 from dy/dt, and (c) changing the prey carrying capacity parameter

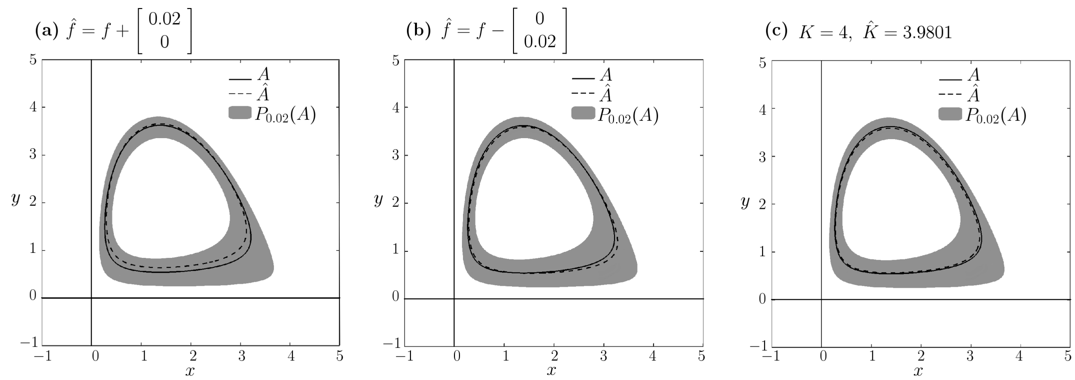

Figure 5.1: Continuation of a predator-prey limit cycle through three pertubations

from $K = 4$ to $K = 0.39801$. (This parameter change perturbs the vector field by less than 0.02 in a neighborhood of the attractor block $\overline{P_{0.02}(A)}$.) The original cycle A is shown with a solid line, the perturbed attractor $\hat{A}$ is shown with a dashed line, and their common attractor block $\overline{P_{0.02}(A)}$ is shaded in grey. //

Example 5.15 highlights a strength of Theorem 5.14: it guarantees attractor continuation across a variety of perturbation forms. However, other methods might be more appropriate when a specific perturbation is of interest. For example, bifurcation analysis in the parameter K would reveal the particular impact of changing prey carrying capacity on the cycle.

The next example demonstrates that attractors may continue for vector field perturbations that *exceed* intensity of attraction.

Example 5.16. Let $x' = f(x) = x(1-x)$. Then $A = \{1\}$ is an attractor for f with intensity $\mu(A) = 0.25$. The set $B = \left[\frac{1}{2}, \frac{3}{2}\right]$ is an attractor block associated with A. Consider a second system $x' = \hat{f}(x) = cx(1-x)$ with $c > 0$. B remains an attractor block associated with $\{1\}$ for any such system. Yet the distance $\left\|f - \hat{f}\right\|_{\text{sup}}$ may exceed 0.25 by taking c sufficiently large. Thus the attractor $\{1\}$ continues immediately to $\{1\}$ despite $\left\|f - \hat{f}\right\|_{\text{sup}}$ exceeding its original intensity. //

Example 5.16 does not contradict Theorem 5.14, but indicates that intensity does not give an upper bound on attractor continuation distance.

5.4 Upper Semicontinuity of Attractors

Upper semicontinuity of attractors means roughly that they cannot expand discontinuously in response to changes in the vector field. Formally,

Definition 5.17. An attractor A corresponding to the vector field f is upper semicontinuous if for any neighborhood V of A there exists a $\delta > 0$ such that any for any second vector field $\hat{f}$ with $\left\| f - \hat{f} \right\|_{\text{sup}} < \delta$, A continues to an attractor $\hat{A} \subset V$.

Proposition 4.9 can be used to prove that attractors are upper semicontinuous in our setting. We establish a technical lemma, clear some debts by proving Propositions 4.9 and 4.11, then demonstrate how Proposition 4.9 implies upper semicontinuity of attractors.

Lemma 5.18. *If A is an attractor, C is a forward invariant compact neighborhood of A, and $\omega(C) = A$, then for any neighborhood V of A there exists a $t \geq 0$ such that $\varphi^t(C) \subset int(V)$.*

Proof: Assume for the sake of contradiction that $\varphi^t(C) \not\subset \text{int}(V)$ for all $t \geq 0$. Let $\mathscr{C} = \{\varphi^t(C) \cap (\text{int}(V))^C \mid t \geq 0\}$ be the collection of nonempty "spillovers" of forward images of C beyond $\text{int}(V)$. Forward invariance of C implies that the collection $\mathscr{C}$ has the finite intersection property: if $S_1, \ldots, S_n \in \mathscr{C}$, then

$$\bigcap_{i=1,\ldots,n} S_i = \bigcap_{i=1,\ldots,n} \left(\varphi^{t_i}(C) \cap (\text{int}(V))^C\right) \tag{5.2}$$

$$= (\text{int}(V))^C \cap \bigcap_{i=1,\ldots,n} \varphi^{t_i}(C) \tag{5.3}$$

$$= (\text{int}(V))^C \cap \varphi^{\max\{t_i\}}(C) \tag{5.4}$$

where the final equality follows from forward invariance of C. We recognize the last set as an element of $\mathscr{C}$, implying that the intersection of finitely many elements of $\mathscr{C}$ is nonempty. Since the elements of $\mathscr{C}$ are each subsets of the compact set C, we conclude that the infinite intersection $\bigcap_{S\in\mathscr{C}} S$ is nonempty. We also have that

$$
\begin{aligned}
A \cap (\mathrm{int}(V))^C &= \omega(C) \cap (\mathrm{int}V)^C && (5.5)\\
&= \left(\bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} \varphi^t(C)} \right) \cap (\mathrm{int}(V))^C && (5.6)\\
&\supset \left(\bigcap_{T\geq 0} \varphi^T(C) \right) \cap (\mathrm{int}(V))^C && (5.7)\\
&= \bigcap_{T\geq 0} \left(\varphi^T(C) \cap (\mathrm{int}(V))^C \right) && (5.8)\\
&= \bigcap_{S\in\mathscr{C}} S && (5.9)
\end{aligned}
$$

On the one hand, we have deduced that $\bigcap_{S\in\mathscr{C}} S \neq \emptyset$, which by 5.5-5.9 implies that $A \cap (\mathrm{int}(V))^C \neq \emptyset$. However, V is a neighborhood of A, implying that $A \subset \mathrm{int}(V)$. This contradiction completes the proof. $\square$

Proof of Proposition 4.9: Property (1) is a direct instance of Lemma 4.8. Towards property (3), fix a neighborhood V of A. Lemma 5.7 gives existence of an attractor block B associated with A. By Lemma 5.18, $\varphi^t(B) \subset \mathrm{int}(V)$ for some $t \geq 0$. Let $K = \varphi^t(B)$. A second application of Lemma 5.18 gives that $\varphi^\tau(K) \subset \mathrm{int}(K)$ for some $\tau > 0$. Let $\delta_1 = \mathrm{dist}(K, \overline{V^C})$ and $\delta_2 = \mathrm{dist}(\varphi^\tau(K), \overline{K^C})$. Let $\delta = \min\{\delta_1, \delta_2\}$ and note $\delta > 0$. By Lemma 3.3 there exists an $r > 0$ such that for all $x \in \mathbb{R}^n$ and all $s \in [0, \tau]$, $\|g\|_\infty < r$ ensures $\left\|\varphi_g^s(x) - \varphi^s(x)\right\| < \delta/2$. Then $\|g\|_\infty < r$ implies that

$$\varphi_g^s(K) \subset \mathcal{N}_{\delta/2}(\varphi^t(K)) \subset \mathcal{N}_{\delta/2}(K) \subset V \tag{5.10}$$

where the second inclusion follows from forward invariance of K and the third from the construction of δ. Additionally, at time τ we have

$$\varphi_g^\tau(K) \subset \mathcal{N}_{\delta/2}(\varphi^\tau(K)) \subset K \tag{5.11}$$

with the final inclusion again following from construction of δ. A simple inductive argument based on inclusions 5.10 and 5.11 implies that $\varphi_g^s(K) \subset V$ for all $s \geq 0$. Since $A \subset K$, this implies $\varphi_g^s(A) \subset V$ for all $s \geq 0$. By Definition 4.4 $P_r(A) \subset V$, and property (3) is established.

The inclusion $A \subset \bigcap_{r>0} P_r(A)$ of property 2 is immediate. The reverse inclusion follows from property (3): any point not in A can be excluded from some neighborhood of A and hence from $P_r(A)$ for some $r > 0$. Hence any point in $P_r(A)$ for all $r > 0$ must be in A. This completes the proof. $\square$

Proof of Proposition 4.11: We wish to show that $\mu(A) > 0$ for any attractor A. Lemma 5.7 gives an attractor block B associated with A. One can readily verify that $B \subset \mathcal{D}(A)$, since no point x in compact, forward-invariant B has empty omega limit set. By Lemma 4.9, there exists an $r > 0$ such that $P_r(A) \subset B \subset \mathcal{D}(A)$. Definition 4.10 then implies that $\mu(A) > 0$. $\square$

Theorem 5.19. *Attractors in the present setting are upper semicontinuous.*

Proof: Fix a neighborhood V of an attractor A for $x' = f(x)$. We will show that there exists a δ such that $\left\| f - \hat{f} \right\|_{\text{sup}} < \delta$ implies that A continues immediately to an attractor $\hat{A} \subset V$ for $\hat{f}$. Using Lemma 2.1, let K be a compact set with $A \subset \text{int}(K) \subset K \subset V \cap \mathcal{D}(A)$. Part 3 of Proposition 4.9 gives an $r > 0$ such that $P_r(A) \subset \text{int}(K)$. Then $\overline{P_r(A)} \subset K \subset \mathcal{D}(A)$. Let $\delta = r$. By Lemma 5.11, $\left\| f - \hat{f} \right\|_{\text{sup}} < r$ implies that $\overline{P_r(A)}$ is an attractor block for $\hat{f}$. Let $\hat{A}$ be the omega limit set of $\overline{P_r(A)}$ under $\hat{f}$. Then $\hat{A}$ is an attractor under $\hat{f}$ (Lemma 5.5), is an immediate continuation of A (Definition 5.12), and is contained in V, as desired. $\square$

Chapter 6

Conclusion

We have adapted McGehee's intensity of attraction from the setting of maps on compact metric spaces [16] to flows on $\mathbb{R}^n$. In both settings, one defines intensity by introducing perturbations to a dynamical system and measuring the critical perturbation level necessary to escape a domain of attraction. To translate from discrete to continuous time, we replaced the ϵ-bounded perturbations to a map's image with r-bounded perturbations to a vector field. Chapter 3 showed that multiflows are an appropriate analog to relations for representing multiple perturbed trajectories for a flow. Chapter 4 related continuous-time intensity to problems in resilience quantification. Lastly, Chapter 5 confirmed that the relationship between attractor intensity and continuation distance carries through from the discrete to the continuous setting.

Several avenues for extending this work seem promising. First, the parallels between intensity of attraction in the discrete and continuous settings suggest a relationship between the two quantities. We conjecture that $\mu(A) = \lim_{t\to 0} \mu_t(A)/t$, where μ gives intensity of an attractor under a flow φ (Definition 4.10) and μ_t gives intensity of the attractor under the map φ^t ([16], section 5). Proving this connection would provide theoretical justification for discretizing time in numerical computations of intensity μ.

Indeed, improving reachable set computations will put intensity estimates on firmer ground and allow stronger statements about attractor continuation. Instead of the simple set-valued Euler scheme employed in Example 4.16, one could use a method based on interval arithmetic that provides inner and outer bounds on reachable sets (for example, [9]). Combining lower bounds on intensity with upper bounds on modeling errors could support rigorous conclusions about reality.

Lastly, we hope to apply intensity theory to models of interesting real-world systems. How in practice does one choose the metric to fit an application, and with what consequences? Here we have considered homogenous, translation-invariant metrics, carrying the same information as a norm. However, the relevant scale of perturbations might vary across state space, requiring a metric in its full generality. We further anticipate that in certain settings, one may want to restrict perturbations to a single coordinate while leaving others unperturbed. This could be accomplished with an extended metric. For example, for x, $y \in \mathbb{R}^n$, let

$$d_k(x, y) = \begin{cases} \infty & \text{if } x_j \neq y_j \text{ for some } j \neq k \\ |x_k - y_k| & \text{otherwise.} \end{cases}$$

Whether intensity theory goes through for general or extended metrics remains to be determined.

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